AN

INTRODUCTION

TO THE

DOCTRINE OF FLUXIONS.

WITH FOURTEEN COPPER-PLATES.

BY JOHN ROWE.

THE FOURTH EDITION,

WITH ADDITIONS AND ALTERATIONS.

TO WHICH IS ADDED,

AN

ESSAY ON THE THEORY.

THE WHOLE REVISED, CAREFULLY CORRECTED, AND PREPARED FOR THE PRESS,

BY THE LATE WILLIAM DAVIS,

Editor of the Gentleman's Mathematical Companion, &c.

LONDON:

PRINTED FOR ANNE DAVIS, NO. 2, ALBION BUILDINGS, ALDERSGATE STREET;

AND SOLD BY SHERWOOD AND CO, AND WM. BAYNES, PATERNOSTER ROW; LACKINGTON AND CO, FINSBURY SQUARE; J. NUNN, GREAT QUEEN STREET, LINCOLN'S INN FIELDS; AND J. S. DICKSON, 18, IVY LANE, PATERNOSTER ROW.

1809.

R. WILKS, PRINTER,
89, Chancery-lane.

BC

ADVERTISEMENT.

ANY apology for reprinting the following work may be deemed unneceſſary, ſince it's utility is ſo well known to Mathematicians and Teachers of the Mathematics, as to render any recommendation of it in this place ſuperfluous; ſuffice it to ſay, that from its great ſcarcity, and the difficulty of procuring a copy of it at any price, as well as the numerous enquiries made for it, induced the late Editor of the Gentleman's Mathematical Companion (Wm. Davis) to prepare the following ſheets for the preſs, and in order to render it ſtill more valuable, he has to this fourth Edition added that ſhort but valuable little Eſſay on the Explanation of the Theory of Fluxions, printed for Wm. Innys, and taken notice of by the late celebrated Mathematician Thomas Simpſon, in the Preface to his excellent Treatiſe on Fluxions.

TO

WILLIAM DAVY, ESQ.

ONE OF HIS MAJESTY'S SERJEANTS AT LAW,

THIS

TRACT

IS

INSCRIBED,

BY HIS MOST OBEDIENT

HUMBLE SERVANT,

THE AUTHOR

THE

PREFACE.

Of all the Mathematical Sciences, *The Doctrine of Fluxions* is the moſt extenſive and ſublime. By this, many Difficulties, unſurmountable by any other known Method, are ſolved with uncommon Expedition, Elegance, and Eaſe.

It is a General Way—for determining the *maxima* and *minima* of Quantities; drawing Tangents to Curves, finding their Points of Inflection and Radii of Curvature:—for obtaining the Lengths of curve Lines, the Areas of curvilineal Spaces, the Surfaces and Solidities of concave and convex Bodies: &c.—In a Word, It extends to the inveſtigating the moſt abſtruſe and difficult Problems in the various Branches of mathematical and philoſophical Science.

The *Method of Fluxions* was firſt invented in the Years 1665 and 1666, by that Prince of Mathematicians and Philoſophers the late Sir *Iſaac Newton*, then Mr. *Newton* about 23 Years old*. It was

* He was Born *December* 25, 1642; Knighted in 1705; and Died *March* 20, 1726.

ſoon after communicated to ſome of his Friends; but, he gave no *public* Specimens thereof until the appearance of his immortal *Philosophiæ Naturalis Principia Mathematica* printed in the Year 1687: The celebrated *German*, therefore, Mr. *Godfrey William Leibnitz*, to whom it was hinted in 1676, applied it in the *Acta Eruditorum*, printed at *Leipſic* in 1684, to a few Problems *de Maximis et Minimis* and Tangents to Curves, and claimed the Invention himſelf* ——Their *Notations* indeed are different; and, Quantities being by both, in Effect conſidered as produced by continual Increaſe after the ſame manner as Space is deſcribed by a Body in Motion; inſtead of the *Velocity* with which a Quantity varies or flows at any Point or Term of the Time in which it is ſuppoſed to be generated, called by Sir *Iſaac* a *Fluxion*, *Leibnitz* takes the *Increment*, or little Part generated in an indefinitely ſmall Portion of Time, and calls it a *Differential*†.

Several excellent Treatiſes have been publiſhed on the Subject; but, as they appear not calculated to introduce the young and unaſſiſted *Beginner* into this abſtruſe and difficult Science, in order to his underſtanding them, a plain and eaſy *Introduction* ſeems to be neceſſary; and for that End the following Sheets are chiefly deſigned.

* See *Raphson's History* of *Fluxions*, printed in the Year 1715; or, the *Commercium Epistolicum*, publiſhed by Order of *the Royal Society* in 1722; wherein, Sir *Isaac* is fully proved the Original Inventor of this noble and most delightful Method.

† *Leibnitz* denotes the *Differential* of any variable quantity x by dx; and Sir *Isaac*, generally, for it's *Fluxion* writes $\dot{x}$; but in his *Principia* flowing quantities are expressed by the capital letters, A, B, &c. and their *Fluxions* or *Increments* by the corresponding ſmall letters *a, b*, &c.

This Tract is divided into *three* Parts: the *first* treats of the *Direct* Method of Fluxions; in which from the generated Quantity or *Fluent* being given, we find the *Fluxion*: and the *second* of the *Inverse* Method; wherein, from the *Fluxion* being known, we find the *Fluent**: the *third* contains *miscellaneous Questions* with their *incremental* and *fluxional Solutions*; which could not, with propriety, be inserted in the former Parts; and to some of which there were occasion to refer.

In this *Third Edition*† are many *Additions* and *Alterations*. The *Constructions* are, in general, *New* ‡. In a Word, It contains, perhaps, a variety of Things not to be found in any other Tract on the Subject.

In order to a thorough understanding of this *Introduction*, it is requisite that the Learner be well acquainted with Arithmetic, Algebra, Geometry, Plane-Trigonometry, Conic-Sections, and the Nature of Logarithms. But, as geometrical and algebraical Treatises, in general, give not the Descriptions, and from thence the deduction of the Properties, of some Curves to be found in the following Sheets; nor the Methods of reducing Quantities into Infinite Series, and of Noting their

* The *Direct* Method of Fluxions, as delivered by *Leibnitz* is, by *Foreigners*, called *Calculus Differentialis*; and the *Inverse* Method, *Calculus Integralis*.

† The *First Edition* was printed in the Year 1751; and the *Second*, with *Alterations* and *Additions*, in 1757.—Cuts.

‡ Those in *art.* 35, 39, 78, and 86, may be seen in other Books.

Powers and Roots, neceſſary to be uſed in *fluxional* Tracts; therefore, theſe Deficiencies are herein ſupplied, though they do not immediately relate to the Buſineſs in Hand.

JOHN ROWE.

January 8, 1767.

THE

CONTENTS.

PART I.

PART. II.

PART III.

PART IV.

AN

INTRODUCTION

TO THE

DOCTRINE

OF

FLUXIONS.

PART I.

CHAPTER I.

Of the Principles of Fluxions, and of the New Notation in Algebra.

1. IN this doctrine, quantities are ſuppoſed to be generated by continual increaſe, after the manner of a ſpace which a body in motion deſcribes.

Thus, a line is ſuppoſed to be generated by a point in motion, a ſuperficies by a line, and a ſolid by a ſuperficies.

2. The velocity with which a quantity flows, or is generated, at any particular point or term of it, is called the *Fluxion* of that quantity at that point or term.

Fig. 1. 2. Thus, if we ſuppoſe the indefinite right line AZ, to move with a parallel motion along the axis AX, or, ſo as always to be parallel to its firſt ſituation, and, at the ſame time, a point to move from A along the ſaid line AZ ſo as to generate or always to be in the curve AY; then the velocity with which the end or point A of the line AZ arrives at any point C, or, which is the ſame, the velocity with which the axis flows or is generated at any particular point C, is called the fluxion of the axis at that point; and the velocity with which the point moves along the line AZ at any point B, that is, the velocity with which the ordinate flows or increaſes at any point B, is called the fluxion of the ordinate at that point; alſo, the velocity with which the point generates or moves along the curve at any point B, is called the fluxion of the curve at that point; likewiſe, the velocity, or degree of quickneſs, with which the curvilineal ſpace ACB flows, or is generated by the line AZ at any term CB, is called the fluxion of the ſaid curvilineal ſpace at that term.

3. Now, if the velocity with which any quantity flows, or is generated, be at every point or term the ſame; that is, if it be neither accelerated nor retarded, the fluxion of it will likewiſe be at every point or term the ſame. But if this velocity be continually increaſed or diminiſhed, then there will be a certain degree of velocity, or fluxion, peculiar to every point or term of the thing deſcribed; and the velocity wherewith the ſaid velocity, at any point or term, is either accelerated or retarded, is called the *Fluxion of the Fluxion*, or the *Second Fluxion*. And, again, if this acceleration or retardation be not uniform, but is con-

tinually varying; or the velocity with which the quantity flows does not uniformly increaſe or decreaſe, then the velocity, or degree of ſwiftneſs, with which this acceleration or retardation either increaſes or decreaſes, is called the *Third Fluxion*; and ſo on.

4. The indefinitely ſmall increaſe of a quantity generated in an indefinitely ſmall particle of time, is called the *Increment* of that quantity.

Thus, if we ſuppoſe bc indefinitely near and parallel to the ordinate BC, and Bn parallel to the abſciſs AC, then Cc, or its equal Bn, is called the increment of the abſciſs AC; nb the increment of the ordinate CB; Bb the increment of the curve AB; and CBbc the increment of the curvilineal ſpace ACB. *Fig.* 3. 4.

5. Now, if ed be ſuppoſed indefinitely near and parallel to bc, and br equal and parallel to Bn or cd, then the difference between nb and re is called the *Increment of the Increment*, or the *Second Increment*, that is, the *Increment* of nb, or *Second Increment* of CB. And, again, if sf be ſuppoſed indefinitely near and parallel to ed, and et equal and parallel to br or df, then the difference between the ſecond increment and that of re and ts, is called the *Third Increment* of CB; and ſo on.

6. *Note.* When a quantity, inſtead of increaſing, is continually diminiſhed, then the indefinitely ſmall particles by which it is leſſened, are not, properly, called *Increments*, but *Decrements*. And both *Increments* and *Decrements* are ſometimes called *Moments*.

7. Now if we ſuppoſe the abſciſs, AC, to flow on with an uniform motion, or equal parts of it to

be generated or deſcribed in equal times, its increment will accurately expreſs or be exactly as its fluxion, ſince velocity is always expreſſed by, or is as the ſpace uniformly deſcribed in a given time; and, therefore, if the curve B*b* did exactly coincide with the tangent or right line TBG, it is evident, that then the increments B*b* and *bn* would likewiſe be deſcribed with uniform motions, and the ſame degrees of velocity with which the curve and ordinate reſpectively flow at the point B, that is, the increments B*b* and *bn* would then be accurately as the fluxions of the curve and ordinate at the point B. But, ſince not two points of the curve are coincident with the tangent, and conſequently the velocities with which the increments are generated are continually varying in every point, therefore the increments and fluxions are not in an exact proportion to each other; or, the increments do not accurately meaſure the velocities or fluxions with which they begin to be generated. However, as the point *b* is continually nearer to a coincidence with the tangent GB, the nearer it approaches the point of contact B, ſo, therefore, if we conceive the ordinate *cb* to move back until it coincides with CB, then the very firſt moment before its coincidence, the curve B*b* and right line BG will be infinitely or rather indefinitely near to a coincidence with each other; and conſequently, in that caſe, the increments B*b* and *bn* will come indefinitely near to meaſure the fluxions of the curve and ordinate, or the velocities with which they flow at the point B; or, becauſe the particles of time in which any increments are generated are ſuppoſed to be indefinitely ſmall, and, conſequently, the acceleration or retardation

of the velocities with which they are generated muſt be ſo too; therefore, they are indefinitely near in proportion to the fluxions of the quantities of which they are increments; but, when *Ratios*, from that of equality, are but indefinitely little, or leſs than can be aſſigned, they may be conſidered as equal*. Hence, therefore, the increments may be taken as proportional to, or for the fluxions, in all operations; and, on the contrary, the fluxions for the increments.

8. Thoſe quantities which are ſuppoſed to flow, or to be generated by continual increaſe, as the abſciſs and ordinate of a curve, are called *Fluents*, and *variable* or *flowing* quantities; and thoſe which neither increaſe nor decreaſe, or admit of no variation, as the parameter of a conic ſection, and the diameter of a circle, are called *fixed*, *given*, and *invariable* quantities.

9. The beginning of the alphabet, viz. a, b, c, *&c.* is uſed to expreſs invariable quantities; and the end of it, viz. z, y, x, *&c.* variable or flowing quantities.

10. The fluxion of any variable quantity x, is denoted by $\dot{x}$; its ſecond fluxion, or the fluxion of $\dot{x}$, by $\ddot{x}$; its third fluxion, or the fluxion of $\ddot{x}$, by $\dddot{x}$; and ſo on. Alſo, the moment, increment, or decrement of x, is denoted by x'; its ſecond moment, increment, or decrement, or the moment, increment, or decrement of x', by x''; and ſo on.

11. The fluxions and moments of invariable quantities, viz. of a, b, c, &c. are evidently $= 0$.

12. Thoſe fluents which are generated in the

* This was allowed by the ancient geometricians, *Euclid Archimedes, &c.*

ſame time, or in equal times, or which begin together and end together, are called *contemporary* fluents; and the fluxions of theſe contemporary fluents are called *contemporary* fluxions. Now, it is evident, if two or more of theſe contemporary fluents are always equal, or in any invariable ratio to each other, that their contemporary fluxions will likewiſe be equal, or in the ſame proportion; and that, on the contrary, if two or more of theſe contemporary fluxions are always equal, or in any invariable ratio to each other, their contemporary fluents will likewiſe be equal, or in the ſame proportion.

We come now to find the *Fluxions* of *Fluents*, or the velocities with which flowing quantities increaſe or decreaſe at any points or terms aſſigned; the buſineſs of which is called the *Direct* Method of Fluxions. But, before we proceed, it may, perhaps, be neceſſary to treat of what is called

THE NEW METHOD OF NOTATION IN ALGEBRA*.

13. In ſurds, the index ſhowing the height of the power to which any given quantity is to be raiſed, is here placed as the numerator of a fraction, whoſe denominator is the radical ſign or index ſhowing the root to be extracted †.

Thus, $\sqrt[3]{x^2}$ is expreſſed by $x^{\frac{2}{3}}$, and $\sqrt{a+x}$ by $\overline{a+x}|^{\frac{1}{2}}$.

* This was invented by the late celebrated Dr. Wallis, and first published in his *Arithmetica Infinitorum* in the year 1656.

† This fraction is called the *Index* of the power, and the given quantity itself the *Root* of the power.

Alſo, $\frac{1}{\sqrt[5]{x^3}}$ is expreſſed by $\frac{1}{x^{\frac{3}{5}}}$ or $x^{-\frac{3}{5}}$, and $\frac{1}{\sqrt[3]{a+x}}$ by $\frac{1}{\overline{a+x}|^{\frac{1}{3}}}$ or $\overline{a+x}|^{-\frac{1}{3}}$ *.

Now, in any geometrical progreſſion, whoſe firſt term is *unity*, or 1, if we take an arithmetical progreſſion, whoſe firſt term is 0, and whoſe ſecond term, or common difference, is the index of the quantity in the ſecond term of the geometrical progreſſion; then, the number in any term of the arithmetical progreſſion will be the index of the quantity in the correſponding term of the geometrical progreſſion; and, therefore, the arithmetical mean between the numbers in any two terms of the arithmetical progreſſion will be the index of the geometrical mean between the quantities in the two correſponding terms of the geometrical progreſſion.

Thus, in the geometrical progreſſion $1. x^{\frac{1}{2}}. x^1. x^{\frac{3}{2}}. x^2. x^{\frac{5}{2}}. x^3.$ &c. where the common multiplier is $x^{\frac{1}{2}}$; if we take the arithmetical progreſſion $0. \frac{1}{2}. 1. \frac{3}{2}. 2. \frac{5}{2}. 3.$ &c. wherein the common difference, or quantity added, is $\frac{1}{2}$; the number in either term of the arithmetical progreſſion is the

* The propriety of uſing *negative* indices is evident; for, to divide any power of x by x, is only to leſſen the index of the power by 1; or, to ſubtract the index of the denominator from that of the numerator. Thus, $\frac{x^3}{x^1}$ is $= x^2$; $\frac{x^2}{x^1}$ is $= x^1$ or x; $\frac{x^1}{x^1}$ is $= x^0 = 1$; $\frac{x^0}{x^1}$ is $= x^{0-1}$, that is, $\frac{1}{x}$ is $= x^{-1}$; &c.

index of the quantity in the correſponding term of the geometrical progreſſion; and the arithmetical mean between the numbers in any two terms of the arithmetical progreſſion is the index of the geometrical mean between the quantities in the two correſponding terms of the geometrical progreſſion. For inſtance, $\frac{3}{2}$, the 4th term of the arithmetical progreſſion, is the index of $x^{\frac{3}{2}}$, the 4th term of the geometrical progreſſion; and the arithmetical mean between $\frac{1}{2}$ and $\frac{5}{2}$, the 2d and 6th terms of the arithmetical progreſſion, which is $\frac{3}{2}$, is the index of the geometrical mean between $x^{\frac{1}{2}}$ and $x^{\frac{5}{2}}$, the ſame two terms of the geometrical progreſſion, which is $x^{\frac{3}{2}}$.

Or, in the deſcending geometrical progreſſion, or ſeries, $1.\frac{1}{x^{\frac{1}{2}}}.\frac{1}{x}.\frac{1}{x^{\frac{3}{2}}}.\frac{1}{x^2}.\frac{1}{x^{\frac{5}{2}}}.\frac{1}{x^3}$. &c. that is, $1.x^{-\frac{1}{2}}.x^{-1}.x^{-\frac{3}{2}}.x^{-2}.x^{-\frac{5}{2}}x^{-3}$. &c. where the common diviſor is $x^{\frac{1}{2}}$, if we take the arithmetical progreſſion, or ſeries, $0.-\frac{1}{2}.-1.-\frac{3}{2}.-2.-\frac{5}{2}.-3$. &c. wherein the common difference, or quantity ſubtracted, is $\frac{1}{2}$, the number in any term of the arithmetical ſeries is the index of the quantity in the correſponding term of the geometrical ſeries: thus, $-\frac{3}{2}$, the 4th term of the arithmetical ſeries, is the index of $x^{-\frac{3}{2}}$, the ſame term of the geometrical ſeries; and the arithmetical mean

between any two terms of the arithmetical ſeries, is the index of the geometrical mean between the ſame two terms of the geometrical ſeries; as, for inſtance, the arithmetical mean between $-\frac{1}{2}$ and $-\frac{5}{2}$, the 2d and 6th terms of the arithmetical ſeries, which is $-\frac{3}{2}$, is the index of the geometrical mean between $x^{-\frac{1}{2}}$ and $x^{-\frac{5}{2}}$, the 2d and 6th terms of the geometrical ſeries, which is $x^{-\frac{3}{2}}$.

Hence we may obſerve, that, in the indices of powers, addition has the effect of multiplication on the reſpective roots, and multiplication of involution, and, *e contra*, ſubtraction of diviſion, and diviſion of evolution; or that, in a word, indices of powers are entirely logarithmical with regard to their roots.

So that, $x^2 \times x^{\frac{3}{2}}$ is $= x^{2+\frac{3}{2}} = x^{\frac{7}{2}}$, $\overline{x^3}|^2$ is $= x^{3\times 2} = x^6$, $\frac{x^5}{x^3}$ is $= x^{5-3} = x^2$, $\overline{x^4}|^{\frac{1}{2}}$ is $= x^{\frac{4}{2}} = x^2$; and $x^2 \times x^{-\frac{3}{2}}$ is $= x^{2-\frac{3}{2}} = x^{\frac{1}{2}}$, $\overline{x^2}|^{-3}$ is $= x^{2\times -3} = x^{-6}$, $\overline{x^4}|^{-\frac{1}{2}}$ is $= x^{4\times -\frac{1}{2}} = x^{-2}$: or, univerſally, $x^m \times x^n$ is $= x^{m+n}$, $\overline{x}|^n$ is $= x^{mn}$, $\frac{x^m}{x^n}$ is $= x^{m-n}$, and $\overline{x^m}|^{\frac{1}{n}}$ is $= x^{\frac{m}{n}}$; where note, *m* and *n* repreſent any affirmative or negative whole numbers or fractions whatever*.

If what has been ſaid be duly conſidered, no difficulty in this *new notation* will occur; the knowledge of which is abſolutely neceſſary, in order to the well underſtanding the following pages.

* These indefinite indices were introduced by the great inventor of fluxions.

CHAPTER II.

Of finding the Fluxion of a given Fluent.

RULE I.

14. To find the Fluxion of a Simple Fluent, or, of that wherein there is but One variable Letter or flowing Quantity.

Mark the variable letter or flowing quantity with a dot over it, and you will have the fluxion required.

Thus, the fluxion of $a\,x$ is $= a\dot{x}$.

Fig. 5. For, if we ſuppoſe the variable rectangle AB to be generated by the given line CB ſetting out from the ſituation AD, and moving along with a parallel motion between the parallel and indefinite ſides AC and DB, it is evident the velocity with which the rectangle flows is equal to the generating line CB drawn into the velocity with which the point C generates or moves along the line AC, that is, the fluxion of the rectangle AB is equal to the invariable line CB drawn into the fluxion of the flowing or variable line AC. Therefore, if AD or CB $= a$, and AC or DB $= x$, then the fluxion of the rectangle ax will be $= a \times \dot{x} = a\dot{x}$.

RULE II.

15. To find the Fluxion of the Product of two or more flowing Quantities drawn into each other.

Multiply the fluxion of each quantity ſeparately

by the other, or the product of the reſt of the quantities; and the ſum of theſe products will be the fluxion required*.

Thus, the fluxion of xy is $=\dot{x}y+x\dot{y}$; the fluxion of xyz is $=\dot{x}yz+x\dot{y}z+xy\dot{z}$; and the fluxion of $vxyz$ is $=\dot{v}xyz+v\dot{x}yz+vx\dot{y}z+vxy\dot{z}$.

1°. That the fluxion of xy is $=\dot{x}y+x\dot{y}$, may thus be proved. Suppoſe a line, coincident with the indefinite right line AF, to move with a parallel motion along the indefinite right line AE perpendicular to AF; and, at the ſame time, another coincident with the line AE, to move with a parallel motion along the line AF; and that the ſaid two lines move with ſuch different degrees of velocity as that their points of interſection be always in the curve AI. Through any point B in the curve, draw CH parallel to AF, and DG parallel to AE. Then, by the line moving along the line AE will the curvilineal ſpace AEI and rectangle DE be generated; and by the line moving along the line AF the curvilineal ſpace AFI and rectangle CF will be generated. Now, before the line moving along the line AE arrives at the ſituation CH, it is evident, that the curvilineal ſpace AEI will increaſe ſlower, or flow with a leſs degree of velocity than the rectangle DE, and afterwards faſter, or with a greater degree of velocity; therefore, at the term CB, they *Fig.* 6.

* The common methods of proving the truth of this rule, which are by the aid of increments, were smartly attacked by the late acute Dr Berkeley, Bishop of Cloyne, in a pamphlet called *The Analyst*, printed in the year 1734; but his objections it is presumed, are by no means applicable to the demonstrations here given; on the contrary, it is not doubted, but the reasoning here advanced will be allowed to be scientific, fair, and conclusive.

will flow or increaſe with one and the ſame degree of velocity. So, likewiſe, the curvilineal ſpace AFI, will flow or increaſe ſlower, or with a leſs degree of velocity than the rectangle CF, before the generating line, moving along the line AF, comes to the ſituation DG; and afterwards faſter, or with a greater degree of velocity; and, therefore, at the term DB, they will increaſe or flow with an equal degree of velocity; that is, the fluxion of the curvilineal ſpace AEI is, at the term CB, equal to the fluxion of the rectangle DE at the ſaid term CB; and the fluxion of the curvilineal ſpace AFI is, at the term DB, equal to the fluxion of the rectangle CF at the ſame term DB. But, (*Art.* 14.) the fluxion of the rectangle DE, at the term CB, is equal to CB drawn into the fluxion of AC; and the fluxion of the rectangle CF, at the term DB, is equal to DB drawn into the fluxion of AD. Hence, therefore, the fluxion of the flowing rectangle ACBD (or of the ſum of the two curvilineal ſpaces ACB and ADB) is equal to CB drawn into the fluxion of AC, added to DB drawn into the fluxion of AD; that is, if we put AC or DB $= x$, and AD or CB $= y$, the fluxion of the rectangle xy will be $= \dot{x}y + x\dot{y}$.

2°. And, that the fluxion of xyz is $= \dot{x}yz + x\dot{y}z + xy\dot{z}$, may thus be proved. Let the length, breadth, and depth, of a parallelopipedon, be repreſented by x, y, and z, reſpectively. Now, this parallelopipedon will be equal to three pyramids, whoſe baſes are xy, xz, and yz, and altitudes z, y, and x, reſpectively*: and therefore, (*Art.* 12.) the ſum of the fluxions of theſe pyramids

* 7 E. 12. Corol. 1.

will be equal to the fluxion of the parallelopipedon $x\,y\,z$. Let either of the pyramids be repreſented by AEI or ACB, which ſuppoſe to be generated by the variable plane AF moving with a parallel motion along the indefinite right line AE; and, at the ſame time, let the parallelopipedon ADGE (whoſe face AD or EG is equal and ſimilar to the baſe of the pyramid at the term CB) be generated by the plane AD always coincident with the plane AF. Now, it is plain, that the pyramid will increaſe ſlower, or flow with a leſs degree of velocity, than the parallelopipedon, before the generating planes arrive at the term CB, and afterwards faſter, or with a greater degree of velocity; therefore, at the ſaid term CB, they will flow, or be generated, with the ſame or an equal degree of velocity: but it is likewiſe plain, that the velocity with which the parallelopipedon is generated is equal to its face AD or CB drawn into the velocity of its motion along the line or ſide AE; therefore, the velocity with which the pyramid is generated, is, at the term CB, equal to its baſe CB drawn into the velocity of its motion at the point C, along the ſide AE; that is, the fluxion of the pyramid ACB is equal to its baſe CB drawn into the fluxion of its altitude AC. Hence, therefore, when the baſe is xy and altitude z, the fluxion of the pyramid is $=xy\times\dot{z}=xy\dot{z}$; and when the baſe is xz and altitude y, the fluxion of it is $=xz\times\dot{y}=x\dot{y}z$; and, laſtly, when the baſe is yz, and altitude x, its fluxion is $=yz\times\dot{x}=\dot{x}yz$. Conſequently, $\dot{x}yz+x\dot{y}z+xy\dot{z}$ (which is the ſum of the fluxions of the three pyramids,) is $=$ the fluxion of the parallelopipedon xyz. *Fig.* 7.

3°. And, hence, that the fluxion of $v\,x\,y\,z$ is $=$

$\dot{v}xyz+v\dot{x}yz+vx\dot{y}z+vxy\dot{z}$, may be thus proved. Put $xyz=u$; then $vxyz=vu$; therefore, (*Art.* 12.) the fluxion of u is $=$ the fluxion of xyz, and the fluxion of vu is $=$ the fluxion of $vxyz$; that is, $\dot{u}=\dot{x}yz+x\dot{y}z+xy\dot{z}$, and $\dot{v}u+v\dot{u}=$ the fluxion of $vxyz$. Wherefore, by restitution, or writing xyz for u and $\dot{x}yz+x\dot{y}z+xy\dot{z}$ for $\dot{u}$, we have $\dot{v}u+v\dot{u}=\dot{v}xyz+v\dot{x}yz+vx\dot{y}z+vxy\dot{z}=$ the fluxion of $vxyz$. Hence

16. The fluxion of xx is $=\dot{x}x+x\dot{x}$; the flux- of xxx is $=\dot{x}xx+x\dot{x}x+xx\dot{x}$; the fluxion of $xxxx$ is $=\dot{x}xxx+x\dot{x}xx+xx\dot{x}x+xxx\dot{x}$; that is, the fluxion of x^2 is $=2x\dot{x}$; the fluxion of x^3 is $=3x^2\dot{x}$; the fluxion of x^4 is $=4x^3\dot{x}$. And if m reprefents any affirmative or pofitive whole number, the fluxion of x^m will be $=mx^{m-1}\dot{x}$.

RULE III.

17. To find the Fluxion of a Fraction.

Multiply the denominator into the fluxion of the numerator; from the product of which, fubtract the numerator drawn into the fluxion of the denominator; then, divide the remainder by the fquare of the denominator, and you will have the fluxion of the fraction required.

Thus, the fluxion of $\frac{x}{y}$ is $=\frac{\dot{x}y-x\dot{y}}{y^2}$.

For, put $z=\frac{x}{y}$; then $yz=x$; and the fluxion of this equation (*Art.* 12 and 15.) is $\dot{y}z+y\dot{z}=$

$\dot{x}$: therefore, by tranſpoſition, $y\dot{z}=\dot{x}-\dot{y}z$, and, by diviſion, $\dot{z}=\frac{\dot{x}-\dot{y}z}{y}$; that is, by reſtitution, or writing $\frac{x}{y}$ for z its equal, $\dot{z}=\frac{\dot{x}-\dot{y}\times\frac{x}{y}}{y}=\frac{\dot{x}y-x\dot{y}}{y^2}=$the fluxion of $\frac{x}{y}$.

Alſo, the fluxion of $\frac{1}{x}$ is $=\frac{-\dot{x}}{x^2}$; the fluxion of $\frac{1}{x^2}$ is $=\frac{-2x\dot{x}}{x^4}$; the fluxion of $\frac{1}{x^3}$ is $=\frac{-3x^2\dot{x}}{x^6}$ and the fluxion of $\frac{1}{x^4}$ is $=\frac{-4x^3\dot{x}}{x^8}$.

For,

1°. Put $z=\frac{1}{x}$; then $xz=1$, and the fluxion of xz will be $=$the fluxion of 1, which (*Art.* 11.) is $=0$; that is, (*Art.* 15.) $\dot{x}z+x\dot{z}=0$; therefore, $x\dot{z}=-\dot{x}z$, and $\dot{z}=\frac{-\dot{x}z}{x}$; or, by writing for z its equal $\frac{1}{x}$, $\dot{z}=\frac{-\dot{x}}{x^2}=$ the fluxion of $\frac{1}{x}$.

2°. Put $z=\frac{1}{x^2}$; then, $zx^2=1$; and the fluxion of this equation (*Art.* 11, 15, and 16.) is $\dot{z}x^2+2zx\dot{x}=0$; therefore, $\dot{z}x^2=-2zx\dot{x}$, and $\dot{z}=\frac{-2zx\dot{x}}{x^2}$; or, by writing $\frac{1}{x^2}$ for its value z, $\dot{z}=\frac{-2x\dot{x}}{x^4}=\frac{-2\dot{x}}{x^3}=$the fluxion of $\frac{1}{x^2}$.

3°. Put $z = \frac{1}{x^3}$; then, $z x^3 = 1$; and the fluxion of this equation (*Art.* 11, 15, and 16.) is $\dot{z}x^3 + 3zx^2\dot{x} = 0$; therefore, $\dot{z}x^3 = -3zx^2\dot{x}$, and $\dot{z} = \frac{-3zx^2\dot{x}}{x^3}$; or, by writing $\frac{1}{x^3}$ for z its value, $\dot{z} = \frac{-3x^2\dot{x}}{x^6} = \frac{-3\dot{x}}{x^4} =$ the fluxion of $\frac{1}{x^3}$.

4°. Put $z = \frac{1}{x^4}$; then $zx^4 = 1$, the fluxion of which equation, (*Art*, 11, 15, and 16) is $\dot{z}x^4 + 4zx^3\dot{x} = 0$: therefore $\dot{z}x^4 = -4zx^3\dot{x}$, and $\dot{z} = \frac{-4zx^3\dot{x}}{x^4}$; that is, by reſtitution, or writing $\frac{1}{x^4}$ for z, $\dot{z} = \frac{-4x^3\dot{x}}{x^8} = \frac{-4\dot{x}}{x^5} =$ the fluxion of $\frac{1}{x^4}$.

But, by the *new* method of notation in algebra, (*Art.* 13.) $\frac{1}{x}$ is $= x^{-1}$, $\frac{1}{x^2}$ is $= x^{-2}$, $\frac{1}{x^3}$ is $= x^{-3}$, $\frac{1}{x^4}$ is $= x^{-4}$: alſo, $\frac{-\dot{x}}{x^2}$ is $= -x^{-2}\dot{x}$, $\frac{-2\dot{x}}{x^3}$ is $= -2x^{-3}\dot{x}$, $\frac{-3\dot{x}}{x^4}$ is $= -3x^{-4}\dot{x}$, $\frac{-4\dot{x}}{x^5}$ is $= -4x^{-5}\dot{x}$. Hence,

18. The fluxion of x^{-1} is $= -x^{-2}\dot{x}$, fluxion of x^{-2} is $= -2x^{-3}\dot{x}$, the fluxion of x^{-3} is $= -3x^{-4}\dot{x}$, the fluxion of x^{-4} is $= -4x^{-5}\dot{x}$; and, if m repreſents any negative whole number, the fluxion of x^m will be $= mx^{m-1}\dot{x}$.

Rule IV.

19. To find the fluxion of an expreſſion compounded of different terms or quantities connected together by the ſigns + and —.

Find the fluxion of each term by the preceding rules; which connect together by the ſigns of the reſpective terms: and you will have the fluxion required.

Thus the fluxion of $ax + xy - \frac{x}{y}$ is $= a\dot{x} + \dot{x}y + x\dot{y} - \frac{\dot{x}y - x\dot{y}}{y^2}$.

For, put $ax + xy - \frac{x}{y} = v$; then $ax + xy = v + \frac{x}{y}$. Now, (*art.* 12.) it is evident, that, the Sum of the Fluxions of ax and xy muſt always be equal to the Sum of the Fluxions of v and $\frac{x}{y}$; that is, $a\dot{x} + \dot{x}y + x\dot{y} = \dot{v} + \frac{\dot{x}y - x\dot{y}}{y^2}$: therefore, by tranſpoſition, $a\dot{x} + \dot{x}y + x\dot{y} - \frac{\dot{x}y - x\dot{y}}{y^2} = \dot{v} =$ the Fluxion of $ax + xy - \frac{x}{y}$.

Rule V.

20. To find the Fluxion of any Power of a given Fluent; whether the Index be integral or fractional, affirmative or negative.

Multiply the expreſſion by the Index of the Power; then, ſubtract 1 from the ſaid Index, and multiply the reſulting expreſſion by the Fluxion of the given Fluent, or of the Root of the Power; and you will have the Fluxion required.

Thus, the fluxion of $\overline{x+x^2}|^3$ is $= 3 \times \overline{x+x^2}|^{3-1} \times \overline{\dot{x}+2x\dot{x}} = \overline{x+x^2}|^2 \times \overline{3\dot{x}+6x\dot{x}}$; the fluxion of $\overline{2ax-x^2}|^{\frac{1}{2}}$ is $= \frac{1}{2} \times \overline{2ax-x^2}|^{\frac{1}{2}-1} \times \overline{2a\dot{x}-2x\dot{x}}$ $= \overline{2ax-x^2}|^{-\frac{1}{2}} \times \overline{a\dot{x}-x\dot{x}} =$ (becauſe, *art.* 13, $\overline{2ax-x^2}|^{-\frac{1}{2}}$ is $= \frac{1}{\overline{2ax-x^2}|^{\frac{1}{2}}}$,) $\frac{a\dot{x}-x\dot{x}}{\overline{2ax-x^2}|^{\frac{1}{2}}}$; the Fluxion of $\overline{a+x}|^{-2}$ is $= -2 \times \overline{a+x}|^{-2-1}$ $\times \dot{x} = \overline{a+x}|^{-3} \times -2\dot{x} = \frac{-2\dot{x}}{\overline{a+x}|^3}$; the Fluxion of $\overline{x-a}|^{-\frac{2}{3}}$ is $= -\frac{2}{3} \times \overline{x-a}|^{-\frac{2}{3}-1} \times \dot{x}$ $= -\frac{2}{3} \times \overline{x-a}|^{-\frac{5}{3}} \times \dot{x} = \frac{1}{\overline{x-a}|^{\frac{5}{3}}} \times + \frac{2}{3}\dot{x} =$ $\frac{-2\dot{x}}{3 \times \overline{x-a}|^{\frac{5}{3}}}$. And, univerſally, the Fluxion of $x^{\frac{m}{n}}$ is $= \frac{m}{n} \times x_n^{-1} \times \dot{x} = \frac{m}{n} x^{\frac{m}{n}-1}\dot{x} = \frac{m}{n} = x$

$^{\frac{m-n}{n}}\dot{x}$; where, *note*, either m or n, or both m and n, may be either integral or fractional affirmative or negative; and consequently, the Index $\frac{m}{n}$, expresses, or represents, any affirmative or negative whole-number or fraction whatsoever. For, by *art.* 16 and 18, if m represents any affirmative or negative whole-number, the Fluxion of x^m will be $= m\,x^{m-1}\,\dot{x}$; and, therefore, if we suppose $y = x^{\frac{m}{n}}$, or, which is the same, $y^n = x^m$, and n to be any affirmative or negative whole-number likewise, by *art.* 12. the Fluxion of y^n will be $=$ the Fluxion of x^m; that is, $n\,y^{n-1}\,\dot{y} = m\,x^{m-1}\,\dot{x}$; which equation divided by $n\,y^{n-1}$ makes $\dot{y} = \frac{mx^{m-1}\,\dot{x}}{n\,y_{n-1}}$, that is, by writing $x^{\frac{m}{n}}$ for y it's value, $\dot{y}$

$$= \frac{m\,x^{m-1}\,\dot{x}}{n \times \overline{x^{\frac{m}{n}}}|^{n-1}} = \frac{m}{n} \times \frac{x^{m-1}\,\dot{x}}{x^{\frac{mn-m}{n}}} = \frac{m}{n} \times x^{m-1-\frac{mn-m}{n}}\,\dot{x} = \frac{m}{n}\,x^{\frac{m-n}{n}}\,\dot{x} =$$ the Fluxion of $x^{\frac{m}{n}}$: and that either m or n, or both m and n, may represent any fractions as well as whole-numbers, is plain, since $\frac{m}{n}$ represents the quotient of any whole-number divided by another, and may be taken for a new m or n; and so on *ad infinitum*: therefore, universally, &c.

Rule VI.

21. To find the Fluxion of a Logarithm.

The Fluxion of the Hyperbolic Logarithm of any quantity, is equal to the Fluxion of that quantity, divided by the quantity itself.

Thus, the Fluxion of the Hyp. Log of x is $= \frac{\dot{x}}{x}$. (See *Part* 3. *Quest.* 8.)

Now, as the Hyp. Log. of 10 (*viz.* 2.30258 &c. See *Part* 3. *Quest.* 9.) is to the Common Log. of 10 (*viz.* 1.) so is the Hyp. Log. of any number, x, to the Common Log. of the same number, x; that is, if we put $L = 2.30258$, &c. as $L : 1 ::$ Hyp. Log. of x: Common Log of x. Therefore, (*art.* 12.) $L : 1 ::$ Fluxion of the Hyp. Log. of x: Fluxion of the Common Log. of x.

Hence, $L : 1 :: \frac{\dot{x}}{x} : \frac{\dot{x}}{Lx} =$ the Fluxion of the Common Log. of x; or, because $\frac{1}{L} = 0.43429$ &c. if we put $\frac{1}{L} = M$; then the Fluxion of the Common Log. of $\dot{x}$ (*viz.* $\frac{\dot{x}}{Lx}$) will be $= \frac{\dot{x}}{x} \times M$; that is, the Fluxion of the Hyp. Log. of any number multiplied by (M or) 0.43429 &c. is = the Fluxion of the Common Log. of the said Number.

SCHOLIUM.

22. THOUGH hitherto we have ſuppoſed when one variable quantity in a Fluential* Expreſſion increaſes, that the others, if any, increaſe likewiſe: yet, it often happens, that ſome of them decreaſe while the others increaſe; in which caſe, the Fluxions of the decreaſing are negative with reſpect to thoſe of the increaſing quantities: and therefore, the ſigns of the terms affected with them ought to be changed.

Thus, if whilſt x increaſes y decreaſes, the Fluxion of the rectangle $x\,y$ will be expreſſed by $\dot{x}\,y - x\,\dot{y}$.

For, let the flowing or variable rectangle A B be continually increaſing by the parallel motion of the variable line CB moving from the ſituation AF along the line AE; and be continually diminiſhing by the motion of the variable line DB, parallel to the line AE, and approaching towards it along the line FA; that is, let the ſide AC = DB continually increaſe whilſt the ſide AD = CB continually decreaſes. Now, putting AC = x, and AD = y, it follows from *art.* 15. *dem.* 1°. that the fluxion of the increaſe will be $= \dot{x}\,y$, and that the fluxion of the decreaſe will be = $x\,\dot{y}$: therefore, the fluxion of the decreaſe ſubtracted from the fluxion of the increaſe, leaves the fluxion of the rectangle $x\,y$ or the velocity with which it flows $= \dot{x}\,y - x\,\dot{y}$. *Fig.* 8.

If what has been ſaid be well underſtood, it is hoped, the Learner will meet with few difficulties in the application thereof, to which we now proceed.

* By a fluential expreſſion is meant, that which contains one, two, or more variable quantities.

CHAPTER III.

Of drawing Tangents to Curves.

23. A *Tangent* is a right line which coincides with a curve in a point, and there ſhews its direction, that is, the inclination it bears to the axis or the angle it makes with the Ordinate.

Fig. 9. 10. Thus, if the right line TB coincides with the curve AY at any point B, the ſaid line is a tangent to it at that point: and, becauſe no two lines can coincide unleſs they have the ſame direction, therefore, the direction of the tangent is properly the direction of the curve at the point of contact.

Now, in general, what is requiſite in order to draw the tangent to any point B, is to find the right line CT, called the ſubtangent; or, the diſtance of the point T from the ordinate CB through which the tangent muſt paſs. And to effect this,

24. Let cb be ſuppoſed parallel to the ordinate CB, and Bn equal and parallel to Cc; then, if the line cb be removed towards CB, in a parallel motion, until it coincides with it; the moment before its coincidence, the triangle Bnb will be in its evaneſcent state; or, which is the ſame thing, if the ſaid two lines be ſeparated from their coincidence, the very firſt moment of their ſeparation produces the ſaid triangle in its naſcent ſtate; and in that moment, the line nb, terminated by the line Bn at one end and by the curve at the other, comes indefinitely near to touch the tangent TB produced: conſequently the triangles bnB and BCT come then indefinitely near to ſimilarity, and may be conſidered as ſimilar: wherefore, by

4 E. 6. $bn : nB :: BC : CT$; that is, (putting the abſciſs $AC = x$, the ordinate $CB = y$, $Cc = Bn = x'$, and $bn = y'$,) $y' : x' :: y : \frac{x'y}{y'} = CT$; or, (*Art.* 7.) $\dot{y} : \dot{x} :: y : \frac{\dot{x}y}{\dot{y}} = CT$.

Or, ſuppoſe the indefinite right line AZ to move with a parallel and uniform motion along the axis AX; and, at the ſame time, a point to move from A along the ſaid line AZ with ſuch velocity as always to be in the curve AY: then, when the line AZ comes to the ſituation CB, if the point moving along thereon were to continue on with an uniform motion, and the ſame degree of velocity with which it arrives at B, it is evident it would move along the tangent TB produced; and therefore, when the point A or C arrives at *c*, the point moving along the line AZ would arrive at *m*: and, becauſe velocity is always as the ſpace uniformly deſcribed in a given time, therefore C*c* or B*n* will be as the velocity with which the point A moves along the axis, *nm* as the velocity with which the point moves from B along the line AZ, and B*m* as the velocity with which the point deſcribes the curve at B or moves from B to *m*; that is, C*c* or B*n* will be as the fluxion of the abſciſs AC, *nm* as the fluxion of the ordinate CB, and B*m* as the fluxion of the curve at the point B: but the triangles *m n* B and BCT are ſimilar; therefore, by 4 E. 6 $mn : nB :: BC : CT$; that is, the fluxion of the ordinate : the fluxion of the abſciſs :: the ordinate : the ſubtangent; or, putting

the abſciſs AC $= x$, and the ordinate CB $= y$,)
$\dot{y} : \dot{x} :: y : \frac{\dot{x} y}{\dot{y}} =$ CT; as before.

25. Now, $\frac{\dot{x} y}{\dot{y}}$ is a general expreſſion for the ſubtangent of every curve whoſe abſciſs is x and ordinate y; but, as it is embarraſſed with the fluxions of x and y, ſo the whole buſineſs is to exterminate them: and to do this, put the equation of the curve into fluxions; from which, or from other properties of the curve, find the value of $\dot{x}$ in terms that are all affected with $\dot{y}$, or, of $\dot{y}$ in terms that are all affected with $\dot{x}$; then, if for $\dot{x}$ or $\dot{y}$ we ſubſtitute its value thus found, in this general expreſſion, *viz.* $\frac{\dot{x} y}{\dot{y}}$ we ſhall have the ſubtangent CT in known terms, or free from fluxions; by which the ſought tangent TB, to a given point in the curve, may be eaſily drawn.

26. *Note.* When the fluxions of x and y, or of the abſciſs and ordinate, are negative to each other, that is, when x increaſes if y decreaſes; or *vice versa*; then the general expreſſion $(\frac{\dot{x} y}{\dot{y}})$ for the ſubtangent will be $-\frac{\dot{x} y}{\dot{y}}$; wherein the negative ſign only ſignifies, that the ſubtangent lies on the other ſide of the ordinate with regard to the abſciſs x. But in finding the fluxion of the equation of the curve, the fluxions of x and y muſt not be conſidered as negative to each other, if you

would have the ſought expreſſion for the ſubtangent come out affirmative.

Example I.

27. To draw a Tangent to a Circle*. *Fig.* 11.

Put the radius EA or ED $= a$, abſciſs AC $= x$, and ordinate CB $= y$; then, CD $= 2a - x$. Now, by 35 E. 3. AC $\times$ CD $=$ CB^2, that is, $2ax - x^2 = y^2$; and this equation put into fluxions is $2a\dot{x} - 2x\dot{x} = 2y\dot{y}$; which divided by $2a - 2x$, makes $\dot{x} = \frac{y\dot{y}}{a-x}$; which ſubſtituted for $\dot{x}$ in the general expreſſion for the ſubtangent, (viz. $\frac{\dot{x}y}{\dot{y}}$, *Art.* 25.) makes the ſubtangent CT $=$ $\frac{y^2}{a-x}$ (which, by writing $2ax - x^2$ for its above value, *viz.* y^2, is) $= \frac{2ax - x^2}{a-x} = \frac{CB^2}{EC}$. Wherefore, if the diſtance ſignified by this expreſſion be ſet off from the point C, in the diameter DA produced, we ſhall have the point T, through which the tangent to the point B muſt paſs.

Construction.

Through the point B deſcribe the ſemicircle EBT: then will T be the point from which the

* By the word circle, is generally meant the space bounded by a curve every-where equidistant from a fixed point; but sometimes the curve itself. (See Sir *Isaac Newton's Arithmetica Universalis.*)

the tangent to the point B is to be drawn. For, by 31 E. 3. the angle EBT will be right; and therefore, by 8 E. 6. the triangles ECB and BCT will be similar, and by 4 E. 6. EC : CB : : BC : CT : ∵ $CT = \frac{CB^2}{EC}$.

Example II.

Fig. 12. 28. To draw a *Tangent* to a *Parabola*.

Suppoſe F to be the focus; and PR the parameter, which put $= a$; alſo, put the abſciſs $AC = x$, and ordinate $CB = y$. Now, by a well known property of the curve, $PR \times AC = CB^2$, that is, $ax = y^2$; the Fluxion of which equation is $a\dot{x} = 2y\dot{y}$; therefore, $\dot{x} = \frac{2y\dot{y}}{a}$; which ſubſtituted for $\dot{x}$, makes the general expreſſion for the ſubtangent CT (*viz.* $\frac{\dot{x}y}{\dot{y}}$ *art.* 25.) $= \frac{2y^2}{a} =$ (by ſubſtituting ax for y^2 its value,) $\frac{2\ a\ x}{a} = 2\ x$. So that, the Subtangent CT is double the abſciſs AC; and conſequently, AT is = AC.

Example III.

Fig. 13. 29. To draw a *Tangent* to an *Ellipſis*.

Put the tranverſe diameter $AD = a$, conjugate $NO = b$, abſciſs $AC = x$, and ordinate $CB = y$. Now, by a well known property of the curve,

$AD^2 : NO^2 :: AC \times CD : CB^2$; that is, $a^2 : b^2 ::$ $x \times \overline{a-x} : \frac{b^2}{a^2} \times \overline{ax-x^2} = y^2$; the Fluxion of which equation is $\frac{b^2}{a^2} \times \overline{a\dot{x}-2x\dot{x}} = 2y\dot{y}$; and this divided by $\frac{b^2}{a^2}$ $\times \overline{a-2x}$, gives $\dot{x} = \frac{2a^2 y\dot{y}}{ab^2 - 2b^2 x}$; which substituted for $\dot{x}$ in $\frac{\dot{x}y}{\dot{y}}$ (the general expression for the Subtangent, *art.* 25,) makes the Subtangent CT $= \frac{2a^2 y^2}{ab^2 - 2b^2 x} =$ (by writing $\frac{b^2}{a^2} \times \overline{ax-x^2}$ for y^2 its equal), $\frac{2a^3 b^2 x - 2a^2 b^2 x^2}{a^3 b^2 - 2a^2 b^2 x} = \frac{2ax - 2x^2}{a-2x}$. Whence we may observe, that, AT is $\left(= CT - CA = \frac{2ax-2x^2}{a-2x} - x\right) = \frac{ax}{a-2x} =$ that part of the Subtangent which falls *without* the curve.

Construction.

Make $Cv = CA$; erect the perpendicular Ar, terminated by a right line drawn from E through v; lastly, make $AT = Ar$: then will T be the point from which the Tangent to the Point B must be drawn. For, by 4 E. 6. $EC : Cv :: EA : Ar$, that is, $\frac{1}{2}a - x : x :: \frac{1}{2}a : \frac{ax}{a-2x} = AT$.

Example IV.

30. To draw a *Tangent* to an *Hyperbola*.

Put the transverse diameter $DA = a$, conjugate $NO = b$, abscis $AC = x$, and ordinate $CB = y$. *Fig.* 14.

Now, by a well known property of the curve $DA^2 : NO^2 :: DC \times AC : CB^2$, that is, $a^2 : b^2 :: \overline{a+x} \times x : \frac{b^2}{a^2} \times \overline{ax+x^2} = y^2$; and, by putting both sides of this equation of the curve into Fluxions, we shall have $\frac{b^2}{a^2} \times a\dot{x} + 2x\dot{x} = 2y\dot{y}$; therefore, by division, $\dot{x} = \frac{2a^2 y\dot{y}}{ab^2 + 2b^2x}$; which multiplied by $\frac{y}{\dot{y}}$, or, which is the same, substituted for $\dot{x}$ in $\frac{\dot{x}y}{\dot{y}}$, the general expression for the Subtangent, *art.* 25) makes the Subtangent $CT = \frac{2a^2y^2}{ab^2+2b^2x}$ (which, by writing for y^2 its equal $\frac{b^2}{a^2} \times \overline{ax+x^2}$, is) $= \frac{2ax+2x^2}{a+2x}$. So that, AT, that Part of the Subtangent *without* the curve, is $= \frac{2ax+2x^2}{a+2x} - x = \frac{ax}{a+2x}$.

Construction.

Make $Cv = CA$; erect the perpendicular Ar, terminated by the right line Ev; lastly, make $AT = Ar$: then will T be the point from which the Tangent to the point B must be drawn. For, by 4 E. 6. $EC : Cv :: EA : Ar$, that is, $\frac{1}{2}a + x : x :: \frac{1}{2}a : \frac{ax}{a+2x} = AT$.

Scholium.

31. From the three foregoing Examples, it may be obſerved, that, in the *Parabola* (*fig.* 12) the internal part of the Subtangent, *viz.* the abſciſs AC, is always *equal* to the external part AT: In the *Ellipſis* (*fig.* 13.) the internal part AC is always *leſs* than the external part AT. And, that, in the *Hyperbola* (*fig.* 14.) the internal part AC is always *greater* than the external part AT.

Example V.

32. To draw a *Tangent* to an *Hyperbola* between its Aſymptotes; that is, taking one of its Aſymptotes for an Axis. *Fig.* 15.

Let EZ and ET be the aſymptotes of the hyperbola YAB, whoſe vertex is A; draw AP and BC parallel to the aſymptote EZ; then will AP be the parameter and equal to PE, which put $=a$; EC an abſciſs, which put $=x$; and CB an ordinate, which put $=y$. Now, becauſe when x increaſes y decreaſes; therefore $\dot{x}$ and $\dot{y}$ are negative to each other, and the general expreſſion for the Subtangent is $-\frac{x\dot{y}}{\dot{y}}$; where the negative ſign ſhews that the point T lies on the other ſide of the ordinate CB with regard to E, (*art.* 26.). By the well known property of the curve, EC : EP : : PA : CB, that is, $x : a :: a : y$; $\therefore x = \frac{a^2}{y}$; the Fluxion of which equation is $\dot{x} = \frac{-a^2\dot{y}}{y^2}$; and this value of $\dot{x}$ being ſubſtituted for it in the above general ex-

preſſion for the Subtangent, *viz.* $-\frac{x\dot{y}}{\dot{y}}$, makes the Subtangent CT $= \frac{a^2 y\dot{y}}{y\dot{y}} = \frac{a^2}{y} =$ (by writing xy for its equal a^2,) $\frac{xy}{y} = x$. So that, CT must be $=$CE.

Example VI.

Fig. 16. 33. To draw a *Tangent* to the *Conchoid* of *Nicomedes* *.

Let fall the perpendicular BH on the aſymptote EZ, and draw BC equal and parallel to HE: Put PE$=a$, EA$=b=$FB, EC$=x=$HB, and CB $=y=$EH. Now, by 47 E. 1. $\overline{FB^2-BH^2}|^{\frac{1}{2}}=$HF, that is, $\overline{b^2-x^2}|^{\frac{1}{2}}=$HF; and becauſe the triangles PCB and BHF are ſimilar, by 4 E. 6. BH : HF :: PC : CB, that is, $x : \overline{b^2-x^2}|^{\frac{1}{2}} :: a+x : y$; $\because y = \frac{a+x}{x} \times \overline{b^2-x^2}|^{\frac{1}{2}}$; which is the equation of the curve; the Fluxion of which is $\dot{y} = \frac{x\dot{x} - \overline{a+x}\,\dot{x}}{x^2} \times \overline{b^2-x^2}|^{\frac{1}{2}} - \frac{1}{2} \times \overline{b^2-x^2}|^{-\frac{1}{2}}\, 2x\dot{x} \times \frac{a+x}{x} = \frac{-a\dot{x}}{x^2} \times \overline{b^2-x^2}|^{\frac{1}{2}} - \frac{x\dot{x}}{\overline{b^2-x^2}|^{\frac{1}{2}}} \times \frac{a+x}{x} = \frac{-ab^2\dot{x} - x^3\dot{x}}{x^2 \times \overline{b^2-x^2}|^{\frac{1}{2}}}$; which ſubſtituted

* This Curve is thus generated:—From a fixed Point P, which is called the Pole of the Conchoid, let a number of right lines PA, PB, PD, &c. be drawn, cutting the right line EZ, which is an asymptote to the Curve; and let the distances EA, FB, GD, &c. be made equal to each other, and a line be drawn through the points A, B, D, &c. then will this line be a curve, called by its inventor *Nicomedes*, a *Conchoid*.

$\dot{y}$, in $-\frac{\dot{x}y}{\dot{y}}$, the general expression for the Subtangent when $\dot{x}$ and $\dot{y}$ are negative to each other, (*art.* 26.) makes the Subtangent $CT = \frac{yx^2 \times \overline{b^2 - x^2}}{ab^2 + x^3}$ (which, by substituting for y its equal in the above equation of the curve, is) $= \frac{\overline{a+x} \times \overline{b^2 - x^2} \times x^2}{ab^2x + x^4} = \frac{\overline{a+x} \times \overline{b^2x - x^3}}{ab^2 + x^3}$.

Construction.

Make $Bn = PC$, draw nr parallel to CP, make $Fv = nr$, draw the right line vC, parallel to which draw BT: then will BT be a Tangent to the point B. For, by 4 E. 6. PC : CB : : PE : EF, that is $a+x : y : : a : \frac{ay}{a+x} = EF$; and BC : CP : : $Bn : nr$, that is, $y : a+x : : a+x : \frac{\overline{a+x}^2}{y} = nr = Fv$; ∵ $Ev = (EF + Fv =) \frac{ay}{a+x} + \frac{\overline{a+x}^2}{y}$. Again $vE : EC : : BC : CT$, that is, $\frac{ay}{a+x} + \frac{\overline{a+x}^2}{y} : x : : y : \frac{xy^2 \times \overline{a+x}}{ay^2 + \overline{a+x}^3} = CT$. But, by the equation of the curve, $y^2 = \frac{\overline{a+x}^2}{x^2} \times \overline{b^2 - x^2}$; which substituted for y^2 makes $CT = \frac{\overline{a+x} \times \overline{b^2x - x^3}}{ab^2 + x^3}$.

EXAMPLE VII.

Fig. 17. 34. To draw a *Tangent* to the *Ciſſoid* of *Diocles* *.

Let ABD be the Ciſſoid, whoſe generating ſemicircle is AFE, and aſymptote EZ an indefinite right line perpendicular to the diameter AE. Put the diameter $AE = a$, abſciſs $AC = x$, ordinate $CB = y$. Now, becauſe by the generation of the curve, the arches Eb and Ad are equal, therefore $bh = d$C, and hE = CA; and, becauſe the triangles bhA and BCA are alike, therefore, by 4 E. 6. Ah : hb : : AC : CB; but, by 13 E. 6. Ah : hb : : hb : hE; ∵ hb : hE : : AC : CB; that is, dC : CA : : AC : CB; or (becauſe $CE = a - x$, and by 35 E. 3. $Cd = \overline{AC \times GE}|^{\frac{1}{2}} = \overline{ax - x^2}|^{\frac{1}{2}}$, $\overline{ax - x^2}|^{\frac{1}{2}} : x : : x : y$; ∵ $x^2 = y \times \overline{ax - x^2}|^{\frac{1}{2}}$; the ſquare of which equation divided by x, is $x^3 = ay^2 - xy^2$; which is the equation of the curve; and the Fluxion of this equation is $3x^2\dot{x} = 2ay\dot{y} - \dot{x}y^2 - 2xy\dot{y}$; therefore, by tranſpoſition and diviſion, $\dot{x} = \frac{2ay\dot{y} - 2xy\dot{y}}{3x^2 + y^2}$, which ſubſtituted for $\dot{x}$, makes the Subtangent CT $\left(= \frac{\dot{x}y}{\dot{y}}, \textit{ art. } 25. \right) =$

* This Curve is thus generated :—In the semicircle AFE, make any two arches Ea and Ac, or Eb and Ad, equal to one another; and through the points a, b, d, c, let right lines be drawn perpendicular to the diameter AE, and transverse lines from the point A; then, from A, through the points of intersection ⊙ BD, &c. draw a line A ⊙ B D, &c. and it will be a Curve, called by its inventor *Diocles* a *Cissoid*.

$\frac{2ay^2-2xy^2}{3x^2+y^2}=$ (by writing for y^2 its equal $\frac{x^3}{a-x}$,) $\frac{2ax-2x^2}{3a-2x}$. Whence we may obſerve, that, AT, the Difference between the abſciſs AC and ſubtangent TC, is $=\frac{ax}{3a-2x}$.

Conſtruction.

Biſect the radius eE in g, and the abſciſs AC in f; draw the perpendicular Ar equal to fg; make $rn=$ Af; and on the point n erect the perpendicular nv, terminated by the right line er; laſtly, draw vT equal and parallel to nA: then will T be the point from which the Tangent to the point B muſt be drawn. For, $rn=\text{A}f=\frac{1}{2}x$, and $r\text{A}=\text{A}g-\text{A}f=\frac{3}{4}a-\frac{1}{2}x$; and by 4 E. 6. $r\text{A}:\text{A}e::rn:nv$ or AT; that is, $\frac{3}{4}a-\frac{1}{2}x:\frac{1}{2}a::\frac{1}{2}x:\frac{ax}{3a-2x}=\text{AT}$.

EXAMPLE VIII.

35. To draw a *Tangent* to the *Cycloid**. *Fig.* 18.

Put OA the radius of the generating circle $=a$, abſciſs AC $=x$, ordinate CB $=y$, CG $=s$,

* This curve is thus generated :—Let a circle roll along upon a right line until it performs one revolution, that is, until it measures out a right line equal to its circumference; then, that point in the circle which first touched the right line will describe the curve called a *Cycloid;* which curve, it is supposed, was first invented by Cardinal *Cusanus*; whose works, in which this figure is found, were transcribed by *J. Scoblant,* in the year 1451.

and the arch AG $= z$. Now, by the nature of the curve, CB = CG + arch GA; that is, $y = s+z$; (for, when the femicircle AGF, generating the femicycloid ABD, is in the pofition BRK, the arch BR or GF muft evidently be equal to RD; and the arch RK or BM or GA be equal to RF or tC; but CG is = tB, and therefore Ct = GB: confequently, arch GA = GB; and therefore, &c.) and the Fluxion of this equation of the curve is $\dot{y} = \dot{s} + \dot{z}$. Let cg be fuppofed indefinitely near and parallel to CG, and Gn equal and parallel to Cc; that is, let let G$g = z'$, $gn = s'$, and nG = Cc $= x'$: then, fuppofing Gg to be a little right line perpendicular to the radius GO, the angles gGn and OGC will be equal; for, if to either of them the angle OGn be added, the fum will be a right angle; and therefore, the right angled triangles gnG and OCG are alike: confequently, by 4 E. 6. gn : nG :: OC : CG; that is, $s' : x' :: a-x : s$; ∵ $s' = \frac{a-x}{s}x'$, or (*art.* 7.) $\dot{s} = \frac{a-x}{s}\dot{x}$; and, gG : Gn :: OG : GC; that is, $z' : x' :: a : s$; ∵ $z' = \frac{a}{s}x'$, or, $\dot{z} = \frac{a}{s}\dot{x}$. Now, by fubftituting $\frac{a-x}{s}\dot{x}$ for $\dot{s}$, and $\frac{a}{s}\dot{x}$ for $\dot{z}$, in the above Fluxion of the equation of the curve, we have $\dot{y} = \frac{a-x}{s}\dot{x} + \frac{a}{s}\dot{x} = \frac{2a-x}{s}\dot{x}$; and this fubftituted for $\dot{y}$, makes the Subtangent CT ($= \frac{\dot{x}y}{\dot{y}}$, *art.* 25.) $= \frac{sy}{2a-x}$.

Construction.

Draw the chord or right line AG, and parallel to it draw TB: then will TB be a Tangent to the curve at the point B. For, the triangles FCG and BCT are similar; therefore, by 4 E. 6. FC : CG :: BC : CT, that is, $2a-x : s :: y : \frac{sy}{2a-x} = CT$.

Or,

Draw the right line EG perpendicular to the radius GO and equal to GB; and through the point E draw the right line TB: then will the said line TB be a Tangent to the curve at the point B. For, let gb be supposed indefinitely near and parallel to GB; then, because the arch $AG = GB = gv$, and the arch $AG + Gg = gv + vb$; therefore Gg or $Bv = vb$; consequently, if we suppose Gg to be a little right line perpendicular to the radius OG or coinciding with the right line EG produced, and Bv to be a little right line parallel to it, and also Bb to be a little right line coinciding with a Tangent to the curve at the point B, the triangles bvB and BGE will be similar: therefore, &c.

Example IX.

36. To draw a *Tangent* to the Curve AB, whose Equation (putting $GC = x$, $CB = y$, and $a =$ a given quantity *more* than 1,) is $a^x = y$. Fig. 19.

Put A = the Hyperbolic Logarithm of a, and Y = the Hyperbolic Logarith of y; then, by the

nature of Logarithms, $x\mathrm{A}. = \mathrm{Y}$; the Fluxion which equation is $\dot{x}\mathrm{A} = \dot{\mathrm{Y}} = $ (by *art.* 21.) $\frac{\dot{y}}{y}$; which divided by A, makes $\dot{x} = \frac{\dot{y}}{\mathrm{A}y}$; and this substituted for $\dot{x}$ makes the Subtangent CT ($= \frac{\dot{x}y}{\dot{y}}$, *art.* 25.) $= \frac{y}{\mathrm{A}y} = \frac{1}{\mathrm{A}}$. Whence we may observe, that, the Subtangent being an invariable quantity, the Curve AB is the *Logarithmic* Curve, whose Asymptote is GF *.

37. Hitherto we have treated only of Curves referred to an axis; or, of those whose ordinates are parallel to one another; We shall therefore now proceed to the drawing of Tangents to *Spirals*; or, to those Curves whose ordinates issue from one and the same fixed Point: Where *note*, the ordinate and subtangent are always perpendicular to each other.

Fig. 20. 38. Suppose Cb indefinitely near to CB; that is, let the ∠ BCb be supposed indefinitely small; and with the ordinate CB, as a radius, let the indefinitely small arch Bn be described; which being considered as a little right-line perpendicular to Cb, and the indefinitely small part of the curve Bb as coinciding with a Tangent to it at the point B; then, because the ∠ BCb is indefinitely small,

* This Curve is thus generated:—In the indefinite right line GF, make GK = KC = CF, &c. and the perpendiculars GA, KI, CB, FD, &c. in geometrical proportion continued: then a curve drawn through the points A, I, B, D, &c. will be the *Logarithmic* Curve; which is so called, because the distances GK, GC, GF, &c. being in arithmetical progression, are as the Logarithms of the ordinates KI, CB, FD, &c.

the Δs bnB and BCT will be indefinitely near to ſimilarity; that is, in the very firſt moment of the exiſtence of the $\Delta\, bn$B, the ſaid Δ may be conſidered as ſimilar to the Δ BCT; therefore, if we put the ordinate CB $= y$, B$n = x'$, and $nb = y'$, by 4 E. 6. we ſhall have $y' : x' :: y :$ CT; that is, (*art.* 7.) $\dot{y} : \dot{x} :: y :$ CT; $\because$ CT $= \frac{\dot{x}y}{\dot{y}}$; which is the ſame General Expreſſion for the Subtangent as that before found for curves referred to an Axis, *art.* 25.

Or, Let the indefinite right line CZ turn like the radius of a circle, round the fixed point or centre C with an uniform motion; and, at the ſame time, let a point moving along thereon, generate, or move with ſuch degrees of velocity as always to be in the Curve CBY, to which ſuppoſe the right line TG a Tangent at the point B; alſo, let Bm be a Tangent to the circular arch DBn deſcribed with the ordinate CB as a radius. Now, when the point generating the curve CY arrives at B, if it was to continue on in the ſame direction, with an uniform motion, and the ſame degree of velocity that it arrives thereat, it would move along the Tangent TB produced, or right line BG; which right line would always be as the Fluxion of the Spiral at the point B: So likewiſe, if we ſuppoſe a point to move from B, in the ſame direction, and with the ſame uniform motion, that the point generating the circular arch DBn arrives at B, it would move along the Tangent or right line Bm perpendicular to the ordinate or radius CB; which right line would always be as the Fluxion of the ſaid arch at the point B: And,

ſince the direction of the point moving from C to Z is always perpendicular to that of the point generating the circular arch Dn; therefore, when the point moving from B to G arrives at G, if Gm be parallel to ZC, the point moving along Bm will be arrived at m; and therefore, mG will be as the Fluxion of the ordinate at the point B. Hence, becauſe the $\triangle$s GmB and BCT are ſimilar, by 4 E. 6. Gm : mB :: BC : CT; that is, the Fluxion of the ordinate CB: the Fluxion of the arch DB :: the ordinate CB: the ſubtangent CT; or, (putting the arch DB $= x$, and the ordinate CB $= y$,) $\dot{y} : \dot{x} :: y : \frac{x\dot{y}}{\dot{y}} =$ CT; as before.

Example I.

Fig. 21. 39. To draw a *Tangent* to the *Spiral* of *Archimides* *.

Put the circumference of the generating circle AFA $= a$, and its radius CA $= b$; ordinate CB $= y$, arch ARF $= z$; and with the ordinate CB, as a radius, let the circular arch Bn be deſcribed, which put $= x$. Now, by the generation of the curve, $a : b :: z : y$,

* This Curve is thus generated:—With the radius CA, let the circle AFA be described with an equable motion, or the point A describe equal arches in equal times; and, at the same moment of time that the point A begins to generate the circle, let another point begin to move along the said radius from C towards A, and to pass over it with an uniform motion, and such velocity that it may arrive at A at the very same moment of time that the said radius shall have described the circle, or come to be in its first situation: then will the point moving along the radius CA, generate, or describe, the curve CBA, called a *Spiral*; the invention of which is attributed to *Archimedes*.

or $a : b :: \dot{z} : \dot{y}$, $\because \dot{z} = \frac{a\dot{y}}{b}$. But, it is evident, the velocity of the point generating the circle is to the velocity with which the point B generates the arch B*n* as CF is to CB; that is, $\dot{z} : \dot{x} :: b : y$; $\therefore \dot{z} = \frac{b\dot{x}}{y}$. Hence we have $\frac{b\dot{x}}{y} = \frac{a\dot{y}}{b}$; therefore, $b^2\dot{x} = ay\dot{y}$, and $\dot{x} = \frac{ay\dot{y}}{b^2}$, which fubftituted for $\dot{x}$ in the general expreffion for the Subtangent, $\frac{\dot{x}y}{\dot{y}}$, (*art.* 38.) makes the Subtangent $CT = \frac{ay^2}{b^2} =$ (becaufe $a : b :: z : y$, or $ay = bz$,) $\frac{byz}{b^2} = \frac{yz}{b}$.

Conftruction.

With the ordinate CB, as a radius, defcribe the circular arch BD; and draw the right line CT perpendicular to the radius CF and equal to the arch DB: then will T be the point from which the Tangent to the point B muft be drawn. For, the fectors CFRA and CBD are fimilar; and confequently, CF : FA :: CB : BD; that is, $b : z :: y : BD = \frac{yz}{b} = CT$.

EXAMPLE II.

40. To draw a *Tangent* to the *Logarithmic Spiral* *.

Put the curve C*d*B $= z$, and its correfpondent ordinate CB $= y$. Suppofe the angles *e*CB and *Fig.* 22.

* This Curve is thus generated:—In the circle AFA, whofe centre is C, let any arches AD, DE, EF, FG, GH, &c. be

BCb to be indefinitely ſmall and equal; and with the ordinate CB, as a radius, deſcribe the circular arch Bn. Now, by the generation of the curve, Ce : CB :: CB : CB; and therefore, if we conſider the little parts of the curve, eB and Bb, as indefinitely little right lines, by 6 E. 6. the triangles CeB and CBb will be ſimilar, or the angle made by the curve and ordinate be always the ſame: Conſequently, the increments of the ordinate and curve are always in an invariable ratio to each other, or as two fixed quantities a and b; that is, $nb : b$B :: $a : b$; and therefore, if we ſuppoſe Bn to be a little right line perpendicular to Cb, by 47 E. 1. $bn : n$B :: $a : \overline{b^2 - a^2}|^{\frac{1}{2}}$; that is, (putting B$n = x'$, and $nb = y'$,) $y' : x' :: a : \overline{b^2 - a^2}|^{\frac{1}{2}}$, or, (*art.* 7.) $\dot{y} : \dot{x} :: a : \overline{b^2 - a^2}|^{\frac{1}{2}}$; $\therefore \dot{x} = \dot{y} \times \frac{\overline{b^2 - ^2}|^{\frac{1}{2}}}{a}$; which being ſubſtituted for $\dot{x}$ in the general expreſſion for the Subtangent, viz. $\frac{x\dot{y}}{\dot{y}}$, *art.* 38. makes the ſought Subtangent CT $= y \times \frac{\overline{b^2 - a^2}|^{\frac{1}{2}}}{a}$.

made equal to each other, ; and the right lines Cd, Ce, CB, Cb, Ch, &c. in geometrical proportion continued: then, a line drawn through the points h, b, B, e, d, &c. will be the curve called a *Logarithmic Spiral;* which name is given to it, because the arches AD, AE, AF, &c. being in arithmetical progression, are as the Logarithms of the ordinates Cd, Ce, CB, &c.

Scholium.

As no geometrical series can, strictly speaking, terminate in o: so the *Spiral* B e d. &c. though it continually tends towards the centre C, can never absolutely arrive thereat; but will approach it within any distance that can be assigned.

Scholium.

We may obſerve, that, (becauſe by 47 E. 1. $TB^2 = BC^2 + CT^2 = y^2 + \frac{b^2 - a^2}{a^2} y^2 = \frac{b^2 y^2}{a^2}$,) the Tangent TB is $= y \times \frac{b}{a}$; and conſequently, the Tangent and Subtangent are to each other as b to $\overline{b^2 - a^2}|^{\frac{1}{2}}$. Again, becauſe $nb : bB :: a : b$, that is, *(art.* 7.) $\dot{y} : \dot{z} :: a : b$; therefore, *(art.* 12.) $y : z :: a : b$; or, the ordinate and curve are always in a given or fixed ratio to each other; and therefore $z = y \times \frac{b}{a}$. So that the Tangent TB and Curve CdB are equal; and ſine ∠ BTC : radius $:: a : b$, or, s. ∠ BTC $= a$, and rad. (s. ∠ BCT) $= b$.

Example III*.

41. To draw a *Tangent* to the *Spiral* CHBLA, generated by a point moving uniformly along the ſemicircle CDA, from C to A, while the ſaid ſemicircle makes one uniform revolution round the point C as a center. *Fig.* 23, 24.

Let the point b be ſuppoſed indefinitely near to B; and with the ordinate CB, as a radius, deſcribe the arch D Bn. Put the radius of the generating ſemicircle, EB, $= a$, arch CRB $= v$, arch DB $= x$, ordinate CB $= y$, Bb $= \dot{v}$, and $bn = \dot{y}$. Now, becauſe the circumferences of circles are as

* Invented *Anno* 1756.

their radii, by the generation of the curve $a : 2y :: \dot{v} : \dot{x} = \frac{2y\dot{v}}{a}$; and, becauſe by 20 E. 3. ∠ bEB $= 2$ ∠ BCb, $a : y :: v' : 2\,Bn$, ∵ $Bn = \frac{yv'}{2a}$; but, by 47 E. 1. (ſuppoſing Bb to be a right line, and Bn a little right line perpendicular to Cb.) $Bb^2 - bn^2 = nB^2$, that is, $v'^2 - y'^2 = \frac{y^2 v'^2}{4a^2}$; from which equation we have $v' = \frac{2ay'}{\overline{4a^2 - y^2}|^{\frac{1}{2}}}$; that is, (*art.* 7.) $\dot{v} = \frac{2a\dot{y}}{\overline{4a^2 - y^2}|^{\frac{1}{2}}}$. Hence, $\dot{x} = \frac{2y}{a} \times \frac{2a\dot{y}}{\overline{4a^2 - y^2}|^{\frac{1}{2}}} = \frac{4y\dot{y}}{\overline{4a^2 - y^2}|^{\frac{1}{2}}}$: which ſubſtituted for $\dot{x}$, makes $\frac{\dot{x}y}{\dot{y}}$, the general expreſſion for the Subtangent CT (*art.* 38.) $= \frac{y}{\dot{y}} \times \frac{4y\dot{y}}{\overline{4a^2 - y^2}|^{\frac{1}{2}}} = \frac{4y^2}{\overline{4a^2 - y^2}|^{\frac{1}{2}}}$.

Corollary.

Since $\dot{x} = \frac{2y\dot{v}}{a}$, that is, $x' = \frac{2yv'}{a}$; and $Bn = \frac{yv'}{2a}$: therefore, $x' = 4\,Bn$.

Conſtruction.

Draw the chord BG, which biſect in F; and to the point F draw the right line CF; produce

the chord or ordinate CB to I, making IB = BC; alſo, draw the right line IT, making the angle CIT = the angle CFB: then will T be the point from which the Tangent to the point B muſt be drawn. For, by 31 E. 3. the angle CBG is right, and therefore the right angled triangles FBC and ICT are ſimilar; wherefore, by 4 E. 6. BF : BC :: IC : CT; that is, (becauſe by 47 E. 1. BG = $\overline{GC^2 - CB^2}|^{\frac{1}{2}} = \overline{4a^2 - y^2}|^{\frac{1}{2}}$, and therefore BF $= \frac{1}{2}\overline{4a^2 - y^2}|^{\frac{1}{2}}$,) $\frac{1}{2}\overline{4a^2 - y^2}|^{\frac{1}{2}} : y :: 2y : \frac{4y^2}{\overline{4a^2 - y^2}|^{\frac{1}{2}}} = CT$.

CHAPTER IV.

Of finding the Maxima *and* Minima, *or* Greateſt *and* Leaſt *of variable Quantities.*

42. A *Maximum* is the greateſt magnitude of a variable Quantity, which naturally increaſes before it arrives at that magnitude and as naturally decreaſes afterwards; and a *Minimum* is the leaſt magnitude of a variable Quantity, which naturally decreaſes before it arrives at that magnitude, and afterwards as naturally increaſes. Therefore, at that certain term where a Quantity becomes a Maximum or a Minimum, it can neither be increaſing nor decreaſing; and conſequently, its Fluxion is = 0.

Fig. 25. Thus, in the triangle AEF, let the equal and parallel ſides AC and DB of the inſcribed and flowing rectangle CD be continually increaſing, while the other equal and parallel ſides AD and CB are continually decreaſing; that is, let the ſaid rectangle be increaſed by the motion of the variable and decreaſing ſide CB, and decreaſed by the motion of the variable and increaſing ſide DB. Then, it is evident, that, while the rectangle is increaſing faſter, or with a greater degree of velocity, by the motion of the line CB, than it is decreaſing by the motion of the line DB, it will be continually increaſing; and that, when it is decreaſing faſter, or with a greater degree of velocity, by the motion of the line DB, than it is increaſing by the motion of the line CB, it will be continually decreaſing: therefore, when theſe two degrees of velocity become equal, the rectangle will be a Maximum; and be neither increaſing nor decreaſing; or, the Velocity or Fluxion with which it is decreaſing ſubtracted from the Velocity or Fluxion with which it is increaſing, leaves the Velocity or Fluxion with which it flows = o. And, it being manifeſt, that, when the rectangle CD becomes a Maximum, the ſum of the triangles FDB and BCE will be a Minimum; therefore, when the ſum of the ſaid triangles is a Minimum, the Fluxion of the increaſing triangle FDB will be equal to the Fluxion of the decreaſing triangle ECB; and therefore, the Fluxion of the latter ſubtracted from that of the former, gives the Fluxion of the Minimum = o.

43. When any Quantity is a Maximum or a Minimum, all the Affirmative Powers of it will

be ſo too: as will alſo the Sum, Remainder, Product or Quotient, ariſing from its being added to, ſubtracted from, multiplied or divided by, any invariable or given quantity. But any Negative * Power of a Maximum will be a Minimum; and any Negative Power of a Minimum will be a Maximum.

Thus, (1°.) if $a - b + \frac{c}{d} \times \overline{ax - x^2}|^{\frac{1}{2}}$ be a Maximum, then will $ax - x^2$ be alſo a Maximum. And, (2°.) if $\overline{\frac{a^2c^2}{x^2} + a^2 + c^2 + x^2}|^{-1}$ be a Maximum, then will $\frac{a^2c^2}{x^2} + a^2 + c^2 + x^2$ or $\frac{a^2c^2}{x^2} + x^2$ be a Minimum.

For (1°.) it is evident, that, the greater $ax - x^2$ is, the greater will be $a - b + \frac{c}{d} \times \overline{ax - x^2}|^{\frac{1}{2}}$; and therefore, when One of theſe expreſſions is a Maximum, the other will be a Maximum alſo. And (2°.) ſince (*Art.* 13.) $\overline{\frac{a^2c^2}{x^2} + a^2 + c^2 + x^2}|^{-1}$ is $= \frac{1}{\frac{a^2c^2}{x^2} + a^2 + c^2 + x^2}$, it is plain, that, the leſs the denominator of this fractional expreſſion is, the greater will be the quotient; but the ſaid denominator is evidently the

* By an affirmative power, is meant that whose Index is affirmative; and, by a negative power, that whose Index is negative.

the leaſt poſſible when $\frac{a^2 c^2}{x^2} + x^2$ is a Minimum.— Therefore, &c.

44. WHEN in the expreſſion for the Fluxion of a Maximum or a Minimum there are two or more *fluxionary* letters, and each is contained in both affirmative and negative terms; then, the ſum of the terms affected with either of them will be $= 0$.

Thus, if the Fluxion of a Minimum be $\frac{1}{2} a\dot{x} - \dot{x}y + x\dot{y} - \frac{1}{2} b\dot{y} = 0$; then $\frac{1}{2} a\dot{x} - \dot{x}y = 0$, and $x\dot{y} - \frac{1}{2} b\dot{y} = 0$.

For, let the variable rectangle CD, whoſe diagonal is AB, flow in the triangle EAF. Put FA $= a$, AE $= b$, AC or DB $= x$, and CB or AD $= y$. Then will the triangle FAB be $= \frac{1}{2} a x$, the triangle AEB be $= \frac{1}{2} by$, and the rectangle CD be $= xy$; and therefore, the ſum of the triangles FDB and BCE will be $= \frac{1}{2} a x - xy + \frac{1}{2} by$; which, when the rectangle CD or xy is a Maximum, is evidently a Minimum. Now, it is plain, if x increaſes y muſt decreaſe, and therefore, (*art.* 22.) their Fluxions are negative to each other, and the Fluxion of the ſaid Minimum is $\frac{1}{2}a\dot{x} - \dot{x}y + x\dot{y} - \frac{1}{2}b\dot{y} = 0$, and the Fluxion of the ſaid Maximum is $\dot{x}y - x\dot{y} = 0$, or, the Fluxion of the triangle ADB or ACB is $\frac{1}{2}\dot{x}y - \frac{1}{2}x\dot{y} = 0$: but, the Fluxions of the triangles FAB and BAE are manifeſtly equal, ſince one increaſes as faſt as the other decreaſes. Hence, therefore, (the Fluxion of the triangle ACB being $= 0$,) the Fluxion of the triangle FAB is $=$ the Fluxion of the triangle ECB; that is, $\frac{1}{2}a\dot{x} = \dot{x}y$, (*art.* 15. *demon.* 1°.) conſe-

quently, $\frac{1}{2}a\dot{x} - \dot{x}y = 0$; and therefore $x\dot{y} - \frac{1}{2}b\dot{y} = 0$.

45. In plane Figures, it is, in effect, the same thing to find the *greatest* Area that can be contained under a *given* Perimeter, as to find a *given* Area under the *least* Perimeter.—It may also be observed, that, to find the *greatest* Solid that can be contained under a *given* Surface, is the same as to find a given Solid under the least Surface.

46. Generally, when a variable quantity admits of a Maximum, its Minimum is Nothing; and, when it admits of a Minimum, its Maximum is Infinite.

Example I.

47. To find the Point C in the given right line AB, where the rectangle of the parts AC and CB is a *Maximum*, or greater than any other rectangle $An \times nB$. *Fig.* 26.

Put AB $= a$, and AC $= x$; then CB $= \overline{a-x}$, and therefore the rectangle AC $\times$ CB $= x \times \overline{a-x} = ax - x^2 =$ a Maximum. Now, the Fluxion of a Maximum being $= 0$, the Fluxion of $ax - x^2$ must be $= 0$; that is, $a\dot{x} - 2x\dot{x} = 0$; which divided by $\dot{x}$, makes $a - 2x = 0$; therefore, $a = 2x$, and $x = \frac{1}{2}a$. Consequently, the rectangle of the parts AC and CB is a Maximum, or the greatest possible, when the said parts are equal.

Or, Put AC $= x$, and CB $= y$; then AC $\times$ CB $= xy =$ a Maximum. Now, it is evident, that, if x increases y must decrease; and therefore the Fluxions of x and y are negative to each other;

and confequently, when xy is a Maximum, the Fluxion of it will be $\dot{x}y - x\dot{y} = 0$: but, it is likewife evident, that, the increment of x is equal to the decrement of y; or, that the velocity with which x increafes is equal to the velocity with which y decreafes; that is, $\dot{x} = \dot{y}$: therefore, by dividing by $\dot{x}$ or $\dot{y}$, or by ftriking both $\dot{x}$ and $\dot{y}$ out, in the above Fluxion of the Maximum xy, we fhall have $y - x = 0$; and therefore $x = y$; as before.

Or, In the variable rectangle Ab, let the fide Cb, which is equal and parallel to the fide AD, always be equal to CB: then, while the rectangle is increafing by the motion of the decreafing fide Cb moving from A towards B, it will be decreafing by the motion of the increafing fide bD moving from F towards A. Now, fince the velocities of the points C and D, or the Fluxions of the fides AC and AD, are always equal, (as they evidently muft be, becaufe the fum of thefe fides is always the fame;) therefore, as long as the fide AC continues lefs than the fide AD or Cb, the rectangle will be continually increafing; and after the fide AC becomes equal to the fide Cb, it will be continually decreafing: therefore, when the rectangle is a Maximum, or when it is neither increafing nor decreafing, the fides AC and Cb are equal to each other, or AC = Cb = CB; as before.

Or, Defcribe the femicircle AbB, and let fall the perpendicular bC. Now, by 35 E. 3. AC × CB = Cb^2; and therefore AC × CB is a Maximum when bC is the radius of the femicircle, or when the point C bifects AB; as before.

Example II.

48. To find the point C in the given right line AB, where $AC^m \times CB^n$ is a *Maximum.*

Put $AB = a$, and $AC = x$; then $CB = a - x$, and $AC^m \times CB^n = x^m \times \overline{a-x}|^n = \overline{x^{\frac{m}{n}} \times \overline{a-x}}|^n$ = a Maximum; therefore, (*art.* 43.) $x^{\frac{m}{n}} \times \overline{a-x}$ = a Maximum; that is, (putting $\frac{m}{n} = e$,) $x^e \times \overline{a-x} = ax^e - x^{e+1}$ = a Maximum; the Fluxion of which is $= 0$; that is, $aex^{e-1}\dot{x} - .\overline{e+1}\,x^e\dot{x} = 0$ Now, by dividing this equation by $x^{e-1}\dot{x}$, we have $ae - .\overline{e+1}\,x = 0$; therefore, $ae = \overline{e+1}\,x$, and $x = \frac{ae}{e+1}$; that is, by reſtitution, or writing $\frac{m}{n}$ for e, $x = \frac{m}{m+n}a$: whence the point C is determined.

Or, Put $AC = x$, and $CB = y$; then $x^m \times y^n =$ a Maximum; the Fluxion of which is $= 0$; that is, (becauſe when x increaſes y decreaſes, or the Fluxions of x^m and y^n are negative to each other,) $mx^{m-1}\dot{x} \times y^n - ny^{n-1}\dot{y} \times x^m = 0$: but, the Fluxions of x and y are evidently equal; therefore, throw both $\dot{x}$ and $\dot{y}$ out; then it will be $mx^{m-1} \times y^n - ny^{n-1} \times x^m = 0$. Now, by dividing this equation by $x^{m-1}\,y^{n-1}$, we have $my - nx = 0$; therefore, $nx = my$, and $m : n :: x : y$. So that the ſegments AC

and CB are in direct proportion to the indices of their powers; and $x = \frac{m}{n}y = \frac{m}{n} \times \overline{a - x} = \frac{ma - mx}{n}$; therefore, $nx = ma - mx$, and $x = \frac{m}{m+n}a$; as before.

EXAMPLE III.

49. To find the *greatest* Cone that can be inscribed in a given Sphere.

Fig. 27. Put AD, the diameter of the Sphere, $= a$; .78539, &c. (the area of a circle whose diameter is 1,) $= c$; and AC, the altitude of the cone, $= x$: then $CD = a - x$. Now, by 35 E. 3. $AC \times CD = CB^2$; that is, $x \times \overline{a - x} = ax - x^2 = CB^2$; therefore, (because the square of the diameter is 4 times the square of the radius,) by 2 E. 12. $4acx - 4cx^2$ = the area of the cone's base; which, by 10 E. 12, drawn into $\frac{1}{3}x$, is $\frac{4}{3}acx^2 - \frac{4}{3}cx^3$ = the cone's solidity, a Maximum: therefore, by *art.* 43. $ax^2 - x^3$ = a Maximum; the Fluxion of which is $= 0$; that is, $2ax\dot{x} - 3x^2\dot{x} = 0$; which divided by $x\dot{x}$, gives $2a - 3x = 0$; therefore, $3x = 2a$, and $x = \frac{2}{3}a$. So that, the cone will be a Maximum, or the greatest possible, when its altitude is = two-thirds of the sphere's diameter.

EXAMPLE IV.

50. To find the internal dimenſions of a cylindrical Cup, whoſe capacity is given $=a$, when the ſaid Cup is made with the *leaſt* poſſible quantity of Silver of a given thickneſs.

Put the diameter $=x$, and .78539, &c. (the area of a circle whoſe diameter is 1;) $=c$; then, by 2 E. 12. $cx^2 =$ the area of the bottom, and therefore $\frac{a}{cx^2} =$ the altitude: but, $4cx =$ the circumference of the bottom, and therefore $4cx \times \frac{a}{cx^2} = \frac{4a}{x} =$ the inſide curve ſuperficies. Hence $cx^2 + \frac{4a}{x} =$ the whole inſide ſuperficies; which, becauſe the quantity of ſilver is the leaſt poſſible, is a Minimum; and therefore its Fluxion is $=0$; that is, $2cx\dot{x} - \frac{4a\dot{x}}{x^2} = 0$; which multiplied by x^2, is $2cx^3\dot{x} - 4a\dot{x} = 0$; and this divided by $2\dot{x}$, is $cx^3 - 2a = 0$; therefore, $cx^3 = 2a$, and $x = \overline{\frac{2a}{c}}\Big|^{\frac{1}{3}}$ $=$ the diameter; which ſubſtituted for x, makes the above $\frac{a}{cx^2} = \frac{a}{c \times \overline{\frac{2a}{c}}\Big|^{\frac{2}{3}}} = \frac{a \times \overline{\frac{2a}{c}}\Big|^{\frac{1}{3}}}{c \times \frac{2a}{c}} = \frac{1}{2} \times \overline{\frac{2a}{c}}\Big|^{\frac{1}{3}} =$

the altitude. So that the diameter is to the altitude as 2 to 1.

Or, Put the diameter $= x$, altitude $= y$, and $.78539$ &c. $= c$; then the whole inside superficies will be $= cx^2 + 4cxy =$ a Minimum; therefore (*art.* 43.) $x^2 + 4xy =$ a Minimum; the Fluxion of which is $= 0$; that is, (supposing x to increase and y to decrease, or the Fluxions of x and y to be negative to each other, as they evidently must be,) $2x\dot{x} + 4\dot{x}y - 4x\dot{y} = 0$. But, $cx^2y = a$, and consequently $x^2y = \frac{a}{c}$; the Fluxion of which equation, (because the Fluxion of $\frac{a}{c}$ is $= 0$; and the Fluxion of y is negative,) is $2x\dot{x}y - x^2\dot{y} = 0$; therefore $x^2\dot{y} = 2x\dot{x}y$; which equation divided by x^2, makes $\dot{y} = \frac{2\dot{x}y}{x}$. Hence, this value of $\dot{y}$ being substituted for it in the above Fluxion of the Minimum, we have $2x\dot{x} + 4\dot{x}y - 8\dot{x}y = 0$; that is, $2x\dot{x} - 4\dot{x}y = 0$; which equation divided by $2\dot{x}$, gives $x - 2y = 0$; consequently $x = 2y$; as before.

Example V.

51. To find the internal dimensions of a Cistern in the form of a rectangular solid, (that is, whose bottom and all four sides are rectangular, when its capacity is $= a$, and it is made with the *least* possible quantity of Lead of a given thickness.

Fig. 28. Put the inside length AC or DB $= x$, breadth AD or CB $= y$, and depth BF or CE $= z$; then,

$xyz = a$, and therefore, $z = \frac{a}{yz}$. Now, the inside superficies of the bottom and four sides is $= AC \times CB + 2AC \times CE + 2CB \times BF = xy + 2xz + 2yz$ $=$ (by substituting $\frac{a}{yz}$ for x,) $\frac{a}{z} + \frac{2a}{y} + 2yz$; which because the cistern is made with the least possible quantity of lead, is a Minimum; and therefore its Fluxion is $= 0$; that is, $-\frac{a\dot{z}}{z^2} - \frac{2a\dot{y}}{y^2} + 2\dot{y}z + 2y\dot{z}$ $= 0$. Now, (*art.* 44.) the sum of the terms affected with $\dot{y}$ is $= 0$, and the sum of the terms affected with $\dot{z}$ is $= 0$; that is, $-\frac{2a\dot{y}}{y^2} + 2\dot{y}z = 0$, and $-\frac{a\dot{z}}{z^2} + 2y\dot{z} = 0$; the former of which equations multiplied by y^2, gives $-2a\dot{y} + 2y^2\dot{y}z = 0$, and this divided by $2\dot{y}$, is $-a + y^2z = 0$, $\because y^2z = a = xyz$, therefore, $y = x$: and the latter of the said equations multiplied by z^2, gives $-a\dot{z} + 2yz^2\dot{z} = 0$; which divided by $\dot{z}$, is $-a + 2yz^2 = 0$, $\because 2yz^2 = a = xyz$, therefore, $2z = x$. Hence, $x = y = 2z$; that is, the length and breadth will be equal, and each equal to twice the depth.

Or, The length, breadth, and depth, being x, y, and z, respectively; and the inside superficies $= xy + 2xz + 2yz =$ a Minimum, as before; if we suppose x and y to increase, then z must necessarily decrease; that is, the Fluxions of x and y being affirmative, the Fluxion of z will be negative; and therefore, the Fluxion of the Minimum will be $\dot{x}y + x\dot{y} + 2\dot{x}z - 2x\dot{z} + 2\dot{y}z - 2y\dot{z} = 0$. But, $xyz = a$, the Fluxion of which equation, (the Fluxion of a being $= 0$, and the Fluxion of z being negative,) is $\dot{x}yz + x\dot{y}z - xy\dot{z} = 0$; $\because \dot{z} = \frac{\dot{x}yz + x\dot{y}z}{xy} = \frac{\dot{x}z}{x} +$

the ſame time: but, the little triangle Bnb is (or may be taken as) ſimilar to the right angled triangle CAB; (for, the arch Bn. being indefinitely ſmall, may be conſidered as a little right line perpendicular to Cb; ſo likewiſe, the angle BCb being indefinitely little, the angle CbA or nbB may be conſidered as equal to the angle CBA; and therefore the angle bBn as equal to the angle BCA;) conſequently, by 4 E. 6. $d : c ::$ CB : BA; but, when the hypothenuſe and perpendicula are d and c, the baſe, by 47 E. 1. will be $\overline{d^2 - c^2}|^{\frac{1}{2}}$. Hence, therefore, $\overline{d^2 - c^2}|^{\frac{1}{2}}$: (CA) $b :: c : \frac{bc}{\overline{d^2 - c^2}|^{\frac{1}{2}}}$ = AB; as before.

Construction.

Through C draw mv parallel to AD, making C$v = d + c$. and C$m = d - c$; deſcribe the ſemicircle mkv; on the interſecting point k erect the perpendicular $kr = c$; and through r draw the right line CB: then will B be the point required. For, by 35 E. 3. $\overline{v\mathrm{C} \times \mathrm{C}m}|^{\frac{1}{2}} = \mathrm{C}k$, that is, $\overline{d^2 - c^2}|^{\frac{1}{2}} = \mathrm{C}k$; and, by 4 E. 6. C$k$: kr :: CA : AB, that is, $\overline{d^2 - c^2}|^{\frac{1}{2}} : c :: b : \frac{bc}{\overline{d^2 - c^2}|^{\frac{1}{2}}}$ = AB.

Example VII.

53. Let the triangle BAD have one angle A in the right line CE: To find a *Maximum* of the Sum of the perpendiculars BC and DE let fall from the other two angles on the right line aforesaid. *Fig.* 30.

Note, AB $= a$, AD $= b$, and the $\angle$ BAD $=$ $90°$.

Put AC $= x$; then, by 47 E. 1. CB $= \overline{BA^2 - AC^2}|^{\frac{1}{2}} = \overline{a^2 - x^2}|^{\frac{1}{2}}$. Now, becauſe the $\angle$ BAD is right, the $\angle$ DAE is the complement of the $\angle$ BAC, and is therefore $=$ $\angle$ ABC; conſequently the triangles BAC and ADE are ſimilar; and therefore, by 4 E. 6. BA : AC :: AD : DE, that is, $a : x :: b : \frac{bx}{a} =$ DE. Hence we have BC $+$ DE $= \overline{a^2 - x^2}|^{\frac{1}{2}} + \frac{bx}{a} =$ a Maximum; the Fluxion of which is $= 0$; that is, $\frac{1}{2} \times \overline{a^2 - x^2}|^{-\frac{1}{2}} \times -2x\dot{x} + \frac{b\dot{x}}{a} = 0$, that is, $-\frac{x\dot{x}}{\overline{a^2 - x^2}|^{\frac{1}{2}}} + \frac{b\dot{x}}{a} = 0$; which multipled by $\overline{a^2 - x^2}|^{\frac{1}{2}} \times a$, gives $-ax\dot{x} + \overline{a^2 - x^2}|^{\frac{1}{2}} b\dot{x} = 0$; therefore, by tranſpoſition and dividing by $\dot{x}$, we have $ax = \overline{a^2 - x^2}|^{\frac{1}{2}} b$; by involution $a^2x^2 = a^2b^2 - b^2x^2$; conſequently, $a^2x^2 + b^2x^2 = a^2b^2$, and $x^2 = \frac{a^2b^2}{a^2 + b^2}$; therefore, $x = \frac{ab}{\overline{a^2 + b^2}|^{\frac{1}{2}}}$, which ſubſti-

tuted for x, makes the Maximum BC + DE ($=$ $\overline{a^2 - x^2}|^{\frac{1}{2}} + \frac{bx}{a}) = \overline{a^2 + b^2}|^{\frac{1}{2}} = \overline{BA^2 + AD^2}|^{\frac{1}{2}}$, that is, by 47 E. 1. BC + DE = the hypothenuſe BD.

Or, Produce BA to b, making bA = AB; draw the right line bD; and perpendicular to CE draw bc: then will the triangles Abc and ABC be equal and ſimilar, for bc is equal and parallel to BC. Now, it is evident, that, the line bD, in every ſituation, will be greater than the ſum of the perpendiculars bc and DE, excepting when it is perpendicular to the line CE; and that then the ſaid perpendiculars will coincide with, and their ſum be equal to, it. Therefore, the Maximum of the ſum of the perpendiculars bc and DE, or of BC and DE, is equal to the ſide bD of the triangle AbD; which, when the $\angle$ bAD or BAD is right, is equal to BD; as before.

Example VIII.

54. To find the *greateſt* right-angled Triangle ACB that can be inſcribed in the given Quadrant AEF; the right angle being formed by the ſine and coſine BC and CA.

Fig. 31. Put the radius or hypothenuſe AB $= a$, and AC $= x$; then, by 47 E. 1. CB $= \overline{a^2 - x^2}|^{\frac{1}{2}}$; therefore, the area of the triangle $= \frac{1}{2}x \times \overline{a^2 - x^2}|^{\frac{1}{2}} = \overline{\frac{1}{4}a^2x^2 - \frac{1}{4}x^4}|^{\frac{1}{2}}$ = a Maximum; conſequently, (*art.* 43.) $a^2x^2 - x^4$ = a Maximum; the Fluxion of which is $= 0$; that is,

$2ax\dot{x} - 4x^3\dot{x} = 0$; which divided by $2x\dot{x}$, makes $a^2 - 2x^2 = 0$; therefore $x^2 = \frac{1}{2}a^2$; consequently AC = CB, and the point B bisects the quadrantal arch EF.

Or, Put AB $= a$, AC $= x$, CB $= y$, EB $= z$; and suppose the point b indefinitely near to B, bc parallel to BC, and Bn equal and parallel to Cc; that is, let B$n = x'$, and B$b = z'$. Now, if we consider the Increment Bb as a little right line perpendicular to the radius AB, the angles nBb and CBA will be equal, and consequently the indefinitely little right angled triangle Bnb will be similar to the right angled triangle BCA; therefore, by 4 E. 6. AB : BC :: bB : Bn, that is, $a : y ::$ $z' : x'$, or (*art.* 7.) $a : y :: \dot{z} : \dot{x}$, $\therefore \dot{z} = \frac{a\dot{x}}{y}$.

But, when the triangle ACB is a Maximum, it is plain, that, AB $\times \frac{1}{2}\dot{z}$ is $=$ BC $\times \dot{x}$; that is, $\frac{1}{2}a\dot{z} = y\dot{x}$, $\therefore \dot{z} = \frac{2y\dot{x}}{a} =$ (by the above.) $\frac{a\dot{x}}{y}$; whence, $2y^2\dot{x} = a^2\dot{x}$, and $2y^2 = a^2 = x^2 + y^2$. Therefore, $x = y$, or AC $=$ CB; as before.

Or, Suppose CD to be always perpendicular to AB. Now, it is evident, the Triangle will be the *greatest* when CD is a Maximum, that is, when it bisects AB, or is the radius of the circumscribing semicircle ACB; and then, it is plain, AC $=$ CB; as before.

EXAMPLE IX.

Fig. 32. 55. In the right line AE, which is perpendicular to the indefinite right line AZ, given AC $= a$, and CE $= b$: To find the point B, where the Angle CBE is a *Maximum*.

Put AE $= a + b = c$, and AB $= x$; then, by 47 E. 1. CB $= \overline{a^2 + x^2}|^{\frac{1}{2}}$, and EB $= \overline{c^2 + x^2}|^{\frac{1}{2}}$. Now, by Trigonometry, CB : radius :: AB : s. ∠ ACB that is, (putting radius $= 1$,) $\overline{a^2 + x^2}|^{\frac{1}{2}} : 1 :: x :$ $\frac{x}{\overline{a^2 + x^2}|^{\frac{1}{2}}} =$ s. ∠ ACB or ECB; and EB : s. ∠ ECB :: EC : s. ∠ CBE, that is, $\overline{c^2 + x^2}|^{\frac{1}{2}} :$ $\frac{x}{\overline{a^2 + x^2}|^{\frac{1}{2}}} :: b : \frac{bx}{\overline{a^2 + x^2}|^{\frac{1}{2}} \times \overline{c^2 + x^2}|^{\frac{1}{2}}} =$ s. ∠ CBE $=$ a Maximum; therefore, (*art.* 43.) $\frac{x^2}{\overline{a^2 + x^2} \times \overline{c^2 + x^2}} = \frac{1}{a^2c^2x^{-2} + a^2 + c^2 + x^2} =$ a Maximum; and $a^2 c^2 x^{-2} + x^2 =$ a Minimum; the Fluxion of which is $= 0$; that is, $-2a^2c^2 x^{-3}\dot{x} + 2x\dot{x} = 0$; and this equation divided by $2x\dot{x}$, makes $-a^2c^2x^{-4} + 1 = 0$; therefore $1 = a^2c^2x^{-4} = \frac{a^2c^2}{x^4}$, and $x^4 = a^2c^2$, and $x = ac^{\frac{1}{2}}$.

So that AB is a geometrical mean between the diſtances AC and AE; and therefore, by 6 E. 6. the triangles CAB and ABE are ſimilar.

*Or**, (in *any* posítition of the line AZ.)—It is evident, that, as long as the angle BCE decreaſes faſter than the angle BEC increaſes, the angle CBE will be increaſing; and that, when the angle BEC increaſes faſter than the angle BCE decreaſes, the angle CBE will be decreaſing: therefore, when the ſaid angle CBE is a Maximum, or when it neither increaſes nor decreaſes, the Fluxions of the angles at C and E will be equal; or, the Increment of the angle BEC will be equal to the Decrement of the angle BCE; that is, if we ſuppoſe the point *b* indefinitely near to B, the $\angle$ BE*b* = $\angle$ BC*b*: therefore, if with EB and CB, as radii, the arches B*m* and B*n* be deſcribed; then, becauſe the circumferences of circles are as their radii, B*m* : B*n* :: EB : CB. Deſcribe the circle BGA, whoſe diameter is AB; and in *fig.* 34. produce EB to G, and BC to I. Draw the right lines GA, AI, and IG. Then, in *fig.* 32. and 33. (by 21 E. 3.) $\angle$ GAI = $\angle$ GBI or EBC; and in *fig.* 34. (by 22 E. 3.) $\angle$ GAI + $\angle$ GBI = 180°. = $\angle$ GBI + $\angle$ EBI; therefore in this *fig.* alſo, $\angle$ GAI = $\angle$ EBI or EBC. Now, if we ſuppoſe the arches B*m* and B*n* to be little right lines perpendicular to E*m* and C*n* reſpectively, then may the triangles *b*B*m* and BAG be conſidered as ſimilar, as may likewiſe the triangles *b*B*n* and BAI; (for, by 31 E. 3. the angles BGA and BIA are right; and, becauſe the points *b* and B are preſumed to be indefinitely near to each other, therefore the $\angle$ E*b*A is indefinitely near to equality with the $\angle$ EBA, and the $\angle$ C*b*A with the $\angle$ CBA; &c.)

Fig. 32. 33. 34.

* Invented Anno 1760.

Example IX.

Fig. 32. 55. In the right line AE, which is perpendicular to the indefinite right line AZ, given $AC = a$, and $CE = b$: To find the point B, where the Angle CBE is a *Maximum.*

Put $AE = a + b = c$, and $AB = x$; then, by 47 E. 1. $CB = \overline{a^2 + x^2}|^{\frac{1}{2}}$, and $EB = \overline{c^2 + x^2}|^{\frac{1}{2}}$. Now, by Trigonometry, CB : radius :: AB : s. ∠ ACB that is, (putting radius $= 1$,) $\overline{a^2 + x^2}|^{\frac{1}{2}} : 1 :: x :$ $\dfrac{x}{\overline{a^2 + x^2}|^{\frac{1}{2}}} =$ s. ∠ ACB or ECB; and EB : s. ∠ ECB :: EC : s. ∠ CBE, that is, $\overline{c^2 + x^2}|^{\frac{1}{2}} :$ $\dfrac{x}{\overline{a^2 + x^2}|^{\frac{1}{2}}} :: b : \dfrac{bx}{\overline{a^2 + x^2}|^{\frac{1}{2}} \times \overline{c^2 + x^2}|^{\frac{1}{2}}} =$ s. ∠ CBE $=$ a Maximum; therefore, (*art.* 43.) $\dfrac{x^2}{\overline{a^2 + x^2} \times \overline{c^2 + x^2}} = \dfrac{1}{\overline{a^2c^2x^{-2} + a^2 + c^2 + x^2}} =$ a Maximum; and $a^2 c^2 x^{-2} + x^2 =$ a Minimum; the Fluxion of which is $= 0$; that is, $-2a^2c^2 x^{-3}\dot{x} + 2x\dot{x} = 0$; and this equation divided by $2x\dot{x}$, makes $-a^2c^2x^{-4} + 1 = 0$; therefore $1 = a^2c^2x^{-4} = \dfrac{a^2c^2}{x^4}$, and $x_4 = a^2c^2$, and $x = ac^{\frac{1}{2}}$.

So that AB is a geometrical mean between the distances AC and AE; and therefore, by 6 E. 6. the triangles CAB and ABE are similar.

*Or**, (in *any* posItition of the line AZ,)—It is evident, that, as long as the angle BCE decreases faster than the angle BEC increases, the angle CBE will be increasing; and that, when the angle BEC increases faster than the angle BCE decreases, the angle CBE will be decreasing: therefore, when the said angle CBE is a Maximum, or when it neither increases nor decreases, the Fluxions of the angles at C and E will be equal; or, the Increment of the angle BEC will be equal to the Decrement of the angle BCE; that is, if we suppose the point b indefinitely near to B, the $\angle$ BEb = $\angle$ BCb: therefore, if with EB and CB, as radii, the arches Bm and Bn be described; then, because the circumferences of circles are as their radii, Bm : Bn :: EB : CB. Describe the circle BGA, whose diameter is AB; and in *fig.* 34. produce EB to G, and BC to I. Draw the right lines GA, AI, and IG. Then, in *fig.* 32. and 33. (by 21 E. 3.) $\angle$ GAI = $\angle$ GBI or EBC; and in *fig.* 34. (by 22 E. 3.) $\angle$ GAI + $\angle$ GBI = 180°. = $\angle$ GBI + $\angle$ EBI; therefore in this *fig.* also, $\angle$ GAI = $\angle$ EBI or EBC. Now, if we suppose the arches Bm and Bn to be little right lines perpendicular to Em and Cn respectively, then may the triangles bBm and BAG be considered as similar, as may likewise the triangles bBn and BAI; (for, by 31 E. 3. the angles BGA and BIA are right; and, because the points b and B are presumed to be indefinitely near to each other, therefore the $\angle$ EbA is indefinitely near to equality with the $\angle$ EBA, and the $\angle$ CbA with the $\angle$ CBA; &c.)

Fig. 32. 33. 34.

* Invented Anno 1760.

wherefore, by 4 E. 6. $bB : BA :: Bm : AG$, and $bB : BA :: Bn : AI$: $\because Bm : Bn :: AG : AI$. Hence $EB : CB :: AG : AI$; and therefore, by 6E. 6. the triangles EBC and GAI are alike, and the $\angle AGI = \angle BEC$: but, by 21 E. 3. the $\angle AGI = \angle ABI$ or ABC; therefore the $\angle BEC = \angle ABC$, and, confequently the triangles CAB and BAE are fimilar; wherefore, by 4 E. 6. $CA : AB :: BA : AE$; $\because AB = \overline{AC \times AE}|^{\frac{1}{2}}$.

Conſtruction.

Produce EA to K, making $AK = AC$; defcribe the femicircle KE; and in *fig.* 33 and 34, draw AQ perpendicular to the diameter EK, and make AB equal to AQ: then will B be the point required. For, by 35 E. 3. $AQ = \overline{KA \times AE}|^{\frac{1}{2}}$, that is, $AB = \overline{AC \times AE}|^{\frac{1}{2}}$.

Example X.

Fig. 35. 56. Given the point C in the radius EA of the circle AB, &c. To find the point B, at which, if a tangent TB and right line CB be drawn to it, the Angle CBT is a *Minimum.*

Draw the radius EB. Now, becaufe by 18 E. 3. the tangent to a circle is always perpendicular to the radius, it is evident, that the angle CBT will be the leaft poffible when the angle CBE is the greateft: It is likewife evident, that, when the faid angle CBE is the greateft poffible, the Fluxions of the angles BCE and BEC will be equal; or, fup-

posing the point b indefinitely near to B, $\angle BCb = \angle BEb$; and therefore, if with CB, as a radius, we describe the little circular arch Bn; then, CB : Bn :: EB : Bb, or, CB : BE :: nB : Bb; and consequently, by 6 E. 6. (because $\angle CBn = \angle EBb$, and therefore $\angle CBE = nBb$,) the triangles CBE and nBb are similar, and $\angle ECB = \angle bnB =$ a right angle. Hence, when the angle CBE is a Maximum, or the angle CBT is a Minimum, the right line CB will be perpendicular to the radius EA.

Or, Because by Trigonometry EB: sine $\angle BCE$ :: EC : sine $\angle CBE$; therefore, the angle CBE will be the greatest when the angle ECB is a right one; but, when the angle CBE is the greatest, the angle CBT is the least possible: therefore, when the angle CBT is a Minimum, the right line CB will be perpendicular to the radius EA; as before.

Example XI*.

57. Given the point C in the radius OA of the semicircle ABE: To find the point B in the said semicircle where the Sum of the right lines CB and EB is a *Maximum*. *Fig.* 36.

Suppose the point b indefinitely near to B; and the little circular arches Bn and Bm, described with CB and EB as radii, to be little right lines perpendicular to the said radii respectively. Now, the angles CBn, OBb, and EBm, being right; therefore $\angle CBO = \angle nBb$, and $\angle OBE = mBb$: but, when CB+EB is a Maximum, it is evident,

* Invented Anno 1761.

that, the Increment nb muſt be $=$ the Decrement bm; and therefore the triangles bnB and bmB are equal and ſimilar. Hence, $\angle nBb$ being $= \angle mBb$, therefore $\angle CBO = \angle OBE = \angle OEB$; and the triangles OCB and BCE are ſimilar; and conſequently, by 4 E. 6. OC : CB :: BC : CE; $\because$ CB $= \overline{CO \times CE}|^{\frac{1}{2}}$.

Conſtruction.

Make CK $=$ CO; deſcribe the ſemicircle KQE, and draw the right line CQ perpendicular to the diameter EK; alſo, make CB $=$ CQ: then will B be the point required. For, by 35 E. 3. CQ $= \overline{EC \times CK}|^{\frac{1}{2}}$, that is, CB $= \overline{EC \times CO}|^{\frac{1}{2}}$.

Corollary.

In order to make the Maximum take place, the given diſtance OC muſt be greater than $\frac{1}{3}$ of the radius OA.

Note.

To find an Algebraical expreſſion for the ſum of the right lines CB and EB when the ſaid ſum is a Maximum. Put the radius EO or OA $= a$, and OC $= b$: then, CB $= \overline{CO \times CE}|^{\frac{1}{2}} = \overline{ab + b^2}|^{\frac{1}{2}}$, and CB : BO :: CE : EB, or EB $= \frac{BO \times CE}{CB}$ $= \frac{a^2 + ab}{\overline{ab + b}|^{\frac{1}{2}}}$: therefore CB + EB $= \overline{ab + b^2}|^{\frac{1}{2}} + \frac{a^2 + ab}{\overline{ab + b^2}|^{\frac{1}{2}}} = \frac{\overline{a+b}|^2}{b^{\frac{1}{2}}.\overline{a+b}|^{\frac{1}{2}}} = \overline{\frac{\overline{a+b}|^3}{b}}|^{\frac{1}{2}}$.

EXAMPLE XII *.

58. Let the given right line CG in its firſt ſituation coincide with the right line AE; and let the end G move along the right line ED in ſuch a manner that the other end C may paſs with an uniform motion from A to E; and, at the ſame time, ſuppoſe a point B to move with the ſame uniform motion from C along the line CG; then, by the motion of this point, will the Curve ABD be deſcribed: To find the point C in the line AE where CF is a *Maximum*; BF being always parallel to DE, and BH parallel to AE. *Fig.* 37.

Put AE $=$ ED $=$ CG $= a$, AF $= x$, and AC $=$ CB $= z$; then, CF $= x-z$, BG $= a-z$, and FE $=$ BH $= a-x$. Now, by 4 E 6. GB : BH :: BC : CF, that is, $a-z : a-x :: z : x-z$; ∵ $ax-az-xz+z^2 = az-xz$, or $ax-2az+z^2=0$; the Fluxion of which equation is $a\dot{x}-2a\dot{z}+2z\dot{z}=0$: but, when CF or $x-z$ is a Maximum, it is evident, that, $\dot{x}=\dot{z}$; therefore, by dividing by $\dot{x}$ or $\dot{z}$, we have $a-2a+2z=0$, or $2z=a$; and therefore $z=\frac{1}{2}a$, or, the point C beſects the line AE when CF is a Maximum.

Corollary.

Since CB $=$ BG; therefore CF $=$ BH $=$ FE $= \frac{1}{2}$CB; and conſequently, when the angle AED is not an Obtuſe but a right one, the angle BCF will, when CF is a Maximum, be $= 60°$.

* Invented Anno 1756.

Example XIII.

Fig. 38. 59. To find the point of Retrogression B in the *Contracted Semicycloid* ABD; whose generating semicircle AGF is greater than its base FD *.

Put the generating semicircle AGF $= a$, base FD $= b$, radius CG $= c$, CG $= s$, OC $= x$, ordinate CB $= y$, and arch AG $= z$; and let the point g be supposed indefinitely near to G, and ng parallel to CF; that is, let nG $= s'$, and Gg $= z'$. Now, if the Increment Gg be supposed a little right line perpendicular to the radius OG, the right angled triangles OCG and Gng will be similar; and therefore, by 4 E. 6. CO : OG :: nG : Gg, that is, $x : c :: s' : z'$, or (*art.* 7.) $x : c :: \dot{s} : \dot{z} = \frac{c\dot{s}}{x}$. By the known property of the curve, $a : b :: z : \text{GB} = \frac{bz}{a}$; therefore CG + GB $= s +$

* This Curve may be thus generated:—Let the semicircle af roll along upon the right line fd equal to it and perpendicular to its diameter fa: then will the curve ABD described by any point A taken *without* the said semicircle and in the radius Oa produced, be a *Contracted Semicycloid*. For, describe the concentric semicircle AGF; in any position of which, as BRK, to the generating point B draw the ordinate CB parallel to the base FD, which said base must evidently be equal and parallel to fd: through the centre o draw MR parallel to the diameter AF; and with the radius oB describe the arch BM: then will the arches AG, MB, and RK, be equal, and tB = CG; and therefore GB = Ct = fr = (by the generation) arch rk; but, semicircle AF : semicircle af :: arch RK : arch rk; therefore, semicircle AF : fd or base FD :: arch AG : GB; which is the property of the *Cycloid*. (See *art.* 35.)

This Curve may, with propriety, be called an *exterior* Cycloid.

$\frac{bz}{a} = y$, which, at the point of Retrogreffion, muft evidently be a Maximum; its Fluxion therefore is $= 0$, that is, because the Fluxions of s and z are negative to each other, $-\dot{s} + \frac{b\dot{z}}{a} = 0$; from which equation we have $\dot{z} = \frac{a\dot{s}}{b}$. Hence $\frac{c\dot{s}}{x} = \frac{a\dot{s}}{b}$; and therefore $ax = bc$, and $x = \frac{bc}{a} = OC$.

Or, * Put the generating femicircle AGF $= a$, bafe FD $= b$, radius OG $= c$, and OC $= x$; and let gb be fuppofed indefinitely near and parallel to GB. Now, it is evident, that, at the point of Retrogreffion B, the tangent muft be perpendicular to the ordinate; that is, at the faid point, the Increment Bb is perpendicular to bv; and therefore, becaufe the $\angle t$Bb = $\angle o$Bv, the right angled triangles Bbv and Bto or GCO are fimilar; and confequently, by 4 E. 6. Bv : vb : : GO : OC. But, by the nature or generation of the curve, a : b : : Gg or Bv : vb. Hence, therefore, $a : b ::$ GO : OC, that is, $a : b :: c : x$; $\because x = \frac{bc}{a} = OC$; as before.

Corollaries.

1. At any point B in the curve, if to the corresponding point G in the circle, we draw the right line TG perpendicular to the radius OG, making TG : GB : : a : b, that is, if TG be made

* Invented Anno 1760.

equal to the arch AG; or if Bx be made equal to the radius Oa, xz be drawn parallel to the tangent GT and equal to the radius OA; then will the right line drawn from T or z to B, be a Tangent to the curve at the point B. For then the triangles TGB or zxB and Bvb will be ſimilar; and therefore, &c.

2. If the right line aQ be drawn perpendicular to the radius OA; then, when the point Q arrives at the baſe FD, the point A will be in the point of Retrogreſſion B. For, ſince at the ſaid point of Retrogreſſion, $x = \frac{bc}{a}$; by analogy, $a : b :: c : x$; that is, ſemicircle BRK : ſemicircle prk :: oR : ot; therefore, the ſemicircles being as their radii oR and or, the point t coincides with r, that is, the ordinate CB coincides with the right line fd; and the triangles Rop and Bot are equal and ſimilar. Conſequently, Rp is perpendicular to po; and therefore, &c.—Hence the following.

Conſtruction.

Make DR = arch ae; draw Rt equal and parallel to Ff, and tB equal and parallel to aQ: then will B be the point of Retrogreſſion required.

CHAPTER V.

Of finding the Points of Inflection in Curves.

60. When a Curve from being Concave becomes Convex towards its axis; or, from being Convex becomes Concave; then, that Point in it where the Change is made, or that which separates the Convex from the Concave part, is called the Point of *Inflection.* So that, if to the Point of Inflection a Tangent be drawn, it will cut the Curve.

Thus, in *fig.* 39. if AB be concave and BY convex, or, in *fig.* 40. if AB be convex and BY concave towards the axis AX; then B is the point of Inflection: where the Tangent TBG cuts the curve. *Fig.* 39. 40.

61. Now, it is evident, that, in any Curve, in order to determine whether the Abscis or Ordinate flows with an accelerated or retarded motion, or, to find the value of its *Second Fluxion*, it is necessary that one of them be made to increase or decrease with a given or uniform motion, with which the swiftness of the increase or decrease of the other may be always compared.

Suppose the Abscis therefore, it being the most natural, always to flow equably, or equal parts of it to be described in equal times; then, because the Direction of the curve from A to B (*fig.* 39.) or from B to Y (*fig.* 40.) continually approaches nearer to a Parallelism with the axis, it is evident,

the ordinate between theſe points muſt flow with a motion continually Retarded: and, becauſe the Direction of the curve from B to Y (*fig.* 39.) or from A to B (*fig.* 40.) approaches continually nearer to a perpendicular to the axis; therefore, between theſe points, the ordinate muſt flow, or increaſe with a motion continually Accelerated. Conſequently, at the point of Inflection, the ordinate will flow with neither an Accelerated nor Retarded but with an Uniform motion: therefore, at the point of Inflection B, the *Second Fluxions* of the abſciſs and ordinate will be = o.

Or, Let B*n* be a given right line always parallel to the baſe, B*m* a Tangent to the curve, and *nm* a right line parallel to the ordinate. Then, it is plain, that, before the ordinate arrives at the point of Inflection, the right line *nm*, in *fig.* 39. will be continually Decreaſing, and afterwards continually Increaſing; or, in *fig.* 40. will be continually Increaſing, and afterwards continually Decreaſing: therefore, at the point of Inflection, it will be neither Increaſing nor Decreaſing; but will, in *fig.* 39. be a Minimum, or, in *fig.* 40. a Maximum; and conſequently, in either caſe, its Fluxion will be = o. But, by *art.* 24. the right lines B*n*, *nm*, and B*m*, are as the Fluxions of the abſciſs, ordinate, and curve, reſpectively. Hence, therefore, the *Second Fluxions* of the abſciſs and ordinate, at the point of Inflection, are = o; as before.

62. Now, ſince, in general, the ordinate CB and the meaſure of its Fluxion, *nm*, flow together; or, ſince the Fluxion of the ordinate CB is always as the right line *nm*, and the ſaid right line is a

Variable or Flowing quantity *; therefore, to find the *Second Fluxion* of a *Fluent*, or the *Fluxion* of an Expression containing the *First Fluxions* of any variable quantities; every *First* Fluxion, not supposed Invariable, must be considered as a distinct Variable quantity, and the Expression be put into Fluxions by the *Rules* laid down in *Chap.* 2.

Thus, the Fluxion of $2y\dot{y}$ is $2\dot{y}\dot{y}+2y\ddot{y}$, that is, $2\dot{y}^2+2y\ddot{y}$; or, if $\dot{y}$ be invariable, it is $2\dot{y}^2$. The Fluxion of $\frac{\dot{x}y}{\dot{y}}$ is $\frac{\ddot{x}y\dot{y}+\dot{x}\dot{y}^2-\ddot{y}\dot{x}y}{\dot{y}^2}$; or, since either $\dot{x}$ or $\dot{y}$ may be supposed invariable, (that is, either x or y to flow with an uniform motion,) if we make $\dot{x}$ invariable, it will be $\frac{\dot{x}\dot{y}^2-\ddot{y}\dot{x}y}{\dot{y}^2}$. Also, the Fluxion of $\frac{x\dot{x}}{\overline{a^2+y^2}|^{\frac{1}{2}}}$ is $\overline{\dot{x}^2+x\ddot{x}}\times\overline{a^2+y^2}|^{-\frac{1}{2}}-\frac{y\dot{y}}{\overline{a^2+y^2}|^{\frac{3}{2}}}\times x\dot{x}$.

* If the right line nm be invariable; then, the curve will degenerate into a right line, and the ordinate will flow uniformly, or the Fluxion of it be always the same.

If the right line nm be variable; then, the velocity with which the ordinate flows will likewise be variable: And, the velocity with which the line nm increases or decreases will always be as the increase or decrease of the velocity with which the ordinate flows; that is, the Fluxion of the line nm will be as the Second Fluxion of the ordinate CB.

If the right line nm does not uniformly increase or decrease; then, the velocity with which the ordinate flows will not uniformly increase or decrease: And, the increase or decrease of the velocity with which the line nm increases or decreases will always be as the increase or decrease of the acceleration or retardation of the velocity with which the ordinate flows; that is, the Second Fluxion of the line nm will be as the Third Fluxion of the ordinate CB.

The Third and Fourth Fluxions, &c. are found in the fame manner, due regard being had to fuch Fluxions as are fuppofed Invariable.

63. Hence, to find the point of Inflection B; put the equation of the curve (where the abfcifs AC is $= x$ or $a \pm x$, and the ordinate CB $= y$,) into Fluxions; from which, or from other properties of the curve, find the value of $\dot{x}$ or $\dot{y}$; and put this $\dot{x}$ or $\dot{y}$ and its value into Fluxions, making both $\ddot{x}$ and $\ddot{y} = 0$: then, by expunging the reft of the Fluxional quantities, you may have x or y, at the point of Inflection fought, determined.

64. But, the point of Inflection may be found without the help of Second Fluxions. For, if TB be a Tangent to it, it is evident, that, AT, the Difference between the fubtangent and abfcifs, will be a *Maximum.* And therefore, to find the point of Inflection, we need only to find a definitive expreffion for the Subtangent, by *Chap.* 3. and the Difference between it and the abfcifs; and then make the Fluxion of this Difference $= 0$.— And fometimes it may be determined in a *New* and different manner; as will be fhewn in the following Example.

EXAMPLE I.

65. To find the point of *Inflexion* B in the *Protracted Semicycloid* ABD; whose generating semicircle AGF is less than its base FD *. *Fig.* 41.

Put the generating semicircle AGF $= a$, base FD $= b$, radius OG $= c$, CG $= s$, OC $= x$, ordinate CB $= y$, and arch AG $= z$; and let the point g be supposed indefinitely near to G, and ng parallel to CF, that is, let G$n = s'$, $ng = x'$, and gG $= z'$. Now, if the Increment Gg be supposed a little right line perpendicular to the radius OG, the right angled triangles OCG and Gng will be similar; and, therefore, by 4 E. 6. OC : CG :: Gn : ng, that is, $x : s :: s' : x'$, or (*art.* 7.) $x : s :: \dot{s} : \dot{x}$, $\therefore \dot{s} = \frac{x\dot{x}}{s}$. By the known property of the curve, $a : b :: z :$ GB $= \frac{bz}{a}$;

* This Curve may be thus generated.—Let the semicircle af roll along upon the right line fd equal to it and pependicular to its diameter fa; then will the curve ABD described by any point A taken *within* the said semicircle and in the radius Oa, be a *Protracted Semicycloid*. For, describe the concentric semicircle AGF; in any position of which, as BRK, to the generating point B draw the ordinate CB parallel to the base FD, which said base must evidently be equal and parallel to fd: through the centre o draw Mr parallel to the diameter af: and with the radius oB describe the arch BM: then will the arches AG, MB, and RK, be equal, and tB $=$ CG; and therefore, GB $=$ Ct $= fr =$ (by the generation) arch rk; but, semicircle AF : semicircle af :: arch RK : arch rk; therefore, semicircle AF : fd or base FD :: arch AG : GB; which is the property of the *Cycloid*. (See *art.* 35.)

This Curve may, with propriety, be called an *interior* Cycloid.

therefore CG + GB $= s + \frac{bz}{a} = y$; the Fluxion of which equation, because the Fluxions of s and z are negative to each other, is $-\dot{s} + \frac{b\dot{z}}{a} = \dot{y}$. Hence, $\dot{y} = -\frac{x\dot{x}}{s} + \frac{b\dot{z}}{a}$, that is, (because by 47 E. 1. $s = \overline{c^2 - x^2}|^{\frac{1}{2}}$,) $\dot{y} = -\frac{x\dot{x}}{\overline{c^2 - x^2}|^{\frac{1}{2}}} + \frac{b\dot{z}}{a}$. But, by 4 E. 6. $s : c :: x' : z'$; that is, $\overline{c^2 - x^2}|^{\frac{1}{2}} : c :: \dot{x} : \dot{z} = \frac{c\dot{x}}{\overline{c^2 - x^2}|^{\frac{1}{2}}}$; wherefore, by substitution, $\dot{y} = (-\frac{x\dot{x}}{\overline{c^2 - x^2}|^{\frac{1}{2}}} + \frac{bc\dot{x}}{a.\overline{c^2 - x^2}|^{\frac{1}{2}}} =) \frac{-ax\dot{x} + bc\dot{x}}{a.\overline{c^2 - x^2}|^{\frac{1}{2}}}$. Now, the Fluxion of this equation, making both $\ddot{x}$ and $\ddot{y} = 0$, is $0 =$

$$\frac{-a\dot{x}^2 \times a.\overline{c^2 - x^2}|^{\frac{1}{2}} + \frac{ax\dot{x}}{\overline{c^2 - x^2}|^{\frac{1}{2}}} \times \overline{-ax\dot{x} + bc\dot{x}}}{a^2.\overline{c^2 - x^2}},$$

that is, (by reduction,) $0 = \frac{-ac^2\dot{x}^2 + bcx^2}{a.\overline{c^2 - x^2}|^{\frac{3}{2}}}$; whence, $0 = -ac + bx$, and therefore $x = \frac{ac}{b} = $ OC.

Or *, Put the generating semicircle AGF $= a$, base FD $= b$, radius OG $= c$, and OC $= x$;

* Invented Anno 1760.

and let gb be ſuppoſed indefinitely near and parallel to GB. Now, it is evident, that, at the point of Inflexion B, the angle made by the tangent and ordinate muſt be a Maximum; that is, at the ſaid point, the $\angle Bbv$ is a Maximum. But, by the nature or generation of the curve, $a : b :: Gg$ or $Bv : vb$; and, by Trigonometry, $bv : vB ::$ s. $\angle vBb$: s. $\angle Bbv$; therefore, the ſaid $\angle Bbv$ is the Greateſt when vB is perpendicular to Bb. Conſequently, at the point of Inflection B, the tangent is coincident with the radius oB; and the triangles Bvb and toB or COG are ſimilar: therefore, by 4 E. 6. $Bv : vb ::$ CO: OG; that is, $a : b :: x : c$; $\because x = \frac{ac}{b} =$ OC; as before.

Corollaries.

1. At any point B in the curve, if to the correſponding point G in the circle, we draw the right line TG perpendicular to the radius OG, making TG : GB $:: a : b$, that is, if TG be made equal to the arch AG; or, if Bx be made equal to the radius Oa, and xz be drawn parallel to the tangent GT and equal to the radius OA; then will the right line drawn from T or z to B be a Tangent to the curve at the point B. For then the triangles TGB or zxB and Bvb will be ſimilar; and, therefore, &c.

2. If the right line AQ be drawn perpendicular to the radius Oa; then, when the point Q arrives at the right line fd parallel to the baſe FD, the point A will be in the point of Inflection B. For, ſince at the ſaid point of Inflexion, $x =$

$\frac{ac}{b}$; by analogy, $b : a :: c : x$, that is, ſemicircle prk : ſemicircle BRK :: oB : ot; therefore, the ſemicircles being as their radii, ro : oB :: Bo : ot; that is, ot o OC is a third proportional to the radii Of and OF, and the triangles roB and Bot are ſimilar. Conſequently, rB is perpendicular to po; and therefore, &c.—Hence the following

Conſtruction.

Make DR = arch fe; draw rM through the point R equal and parallel to fA; make oR = OF; and laſtly, deſcribe the ſemicircles MR and or: then will the interſecting point B of the ſaid ſemicircles be the point of Inflection required.

Example II.

Fig. 42. 66. To find the point of *Inflexion* B in the *Conchoid* of *Nicomedes* AB &c. *

Put PE = a, EA = b, EC = x, and CB = y; then (*art.* 33.) $\dot{y} = \frac{-ab^2\dot{x} - x^3\dot{x}}{x^2 \times \overline{b^2 - x^2}|^{\frac{1}{2}}}$. Now, the Fluxion of this equation, making both $\ddot{x}$ and $\ddot{y} = 0$, is, (by reduction) $0 = \frac{2ab^4x - 3ab^2x^3 - b^2x^4}{x^4 \times \overline{b^2 - a^2}|^{\frac{3}{2}}}\dot{x}^2$; therefore, $2ab^2 - 3ax^2 - x^3 = 0$; by which equation, x, and conſequently the point B, may be determined.—And,

* See the generation of this Curve, *art.* 33. *note.*

if $a = b$, it will be $2a^3 - 3ax^2 - x^3 = 0$; which divided by $a + x$, makes $2a^2 - 2ax - x^2 = 0$; from which quadratic equation we have $x = \overline{3a}|^{\frac{1}{2}} - a$.

Construction.

Make $Pn = \frac{2}{3}PE$; draw the indefinite right line nm perpendicular to Pn; make $Es = \frac{1}{5}EA$; draw the right lines sr and Pm parallel to each other and perpendicular to the right line mr; make $Et = EA$; and parallel to sr draw the right line tC: then an ordinate drawn from the point C will fall on the point of Inflection B. For, by 4. E. 6. $CE : Et :: Pn : nm$, that is, $x : b :: \frac{2}{5}a : \frac{2ab}{5x} = nm$; and $Pn : nm :: mn : nr$, that is, $\frac{2}{5}a : \frac{2ab}{5x} :: \frac{2ab}{5x} : \frac{2ab^2}{5x^2} = nr$; again, $tE : EC :: sE : Er$, that is, $b : x :: \frac{1}{5}b : \frac{1}{5}x := Er$; therefore, ($nE$ being $= \frac{3}{5}a$) $nr = \frac{3}{5}a + \frac{1}{5}x$. Hence, $\frac{2ab^2}{5x^2} = \frac{3}{5}a + \frac{1}{5}x$, and therefore $2ab^2 = 3ax^2 + x^3$, or $2ab^2 - 3ax^2 - x^3 = 0$.

Or,

When $a = b$; make $Pv = PA$, and $PC = Ev$: then will C be the point in the abscis from which the ordinate to the point of Inflection must be drawn. For then $\overline{Pv^2 - PE^2}|^{\frac{1}{2}} = \overline{3a^2}|^{\frac{1}{2}} = Ev = C$; and, therefore, $EC = \overline{3a^2}|^{\frac{1}{2}} - a = x$.

Example III.

67. To find the point of *Inflection* B in the Curve ABY, whose Equation (putting the absciss AC $= x$, ordinate CB $= y$, and the perpendicular AE $= a$,) is $ax^2 = a^2y + x^2y$.

Fig. 43. The Fluxion of the equation of the curve is $2ax\dot{x} = a^2\dot{y} + 2x\dot{x}y + x^2\dot{y}$; therefore, $\dot{y} = \frac{2ax\dot{x} - 2x\dot{x}y}{a^2 + x^2}$, that is, (by writing $\frac{ax^2}{a^2 + x^2}$ for y its value,) $\dot{y} = \frac{2a^3x\dot{x}}{\overline{a^2+x^2}|^2}$; and the Fluxion of this equation, making both $\ddot{x}$ and $\ddot{y} = 0$, is

$$0 = \frac{2a^3\dot{x}^2 \times \overline{a^2+x^2}|^2 - ; 4a^2x\dot{x} + 4x^3\dot{x} \times 2a^3x\dot{x}}{\overline{a^2+x^2}|^4};$$ which multiplied by $\overline{a^2+x^2}|^4$ and divided by $2a^3\dot{x}^2$, gives $0 = \overline{a^2+x^2}|^2 - 4a^2x^2 - 4x^4$; therefore $4x^4 + 4a^2x^2 = \overline{a^2+x^2}|^2$; which equation divided by $x^2 + a^2$, makes $4x^2 = a^2 + x^2$; therefore, $3x^2 = a^2$, and $x = a\sqrt{\frac{1}{3}}$; and, if this value of x be substituted for it in the given equation of the curve, we shall have y, or, the ordinate at the point of Inflection $= \frac{1}{4}a$.

Construction.

Make $Ae = \frac{1}{3}$AE; and with the radius eE describe the arch EC: then will C be the point from which the ordinate to the point of Inflection must

be drawn. For then $eC = \frac{2}{3}a$, and by 47 E. 1. $Ce^2 - eA^2 = AC^2$, that is, $\frac{4}{9}a^2 - \frac{1}{9}a^2 = \frac{1}{3}a^2 = AC^2 = x^2$, or $x = a\sqrt{\frac{1}{3}}$.

Note.

If it were required to find the Asymptote to the curve:—Suppose the abscis's and curve to be indefinitely extended: then, because x^2 will be indefinitely near to equality with $a^2 + x^2$, we shall have y (which by the equation of the curve is $= a \times \frac{x^2}{a^2 + x^2}$,) indefinitely near to equality with the given right line a; that is, y will then be $= a$, *minus* a quantity indefinitely small. Wherefore, if from the point E a right line, EZ, be drawn parallel to the axis, AX, it will be the Asymptote required.

Scholium.

68. In any Curve, in order to know whether it be *concave* or *convex* towards any point assigned in the axis; find the value of $\ddot{y}$ at that point: then, (*art.* 61.) if this value of $\ddot{y}$ comes out Positive, the curve will be *convex* towards the axis; and if it comes out Negative, it will be *concave*.

Thus, in the last Example, if it were required to find whether the Curve be at first *concave* or *convex* towards the axis:—Suppose $a = 10$, and make $x = 1$, or $x =$ any number less than $a\sqrt{\frac{1}{3}}$ or $10\sqrt{\frac{1}{3}}$; then, because $\ddot{y}$ may here be considered to be always as $\overline{a^2 + x^2}|^2 - 4a^2x^2 - 4x^4$, the ex-

preffion for $\ddot{y}$ will come out Pofitive; and therefore, the curve is at firft *convex* towards the axis: And, when $x = 6$, or $x =$ any number greater than $a\sqrt{\frac{1}{3}}$ or $10\sqrt{\frac{1}{3}}$, the faid expreffion will come out Negative; and therefore, then, the curve will be *concave* towards the axis.

CHAPTER VI.

Of finding the Radius of Curvature.

69. As the curvature, or convexity, of all curves, but circles, varies in every point; therefore, if circles are defcribed coinciding with a given curve in any number of points, the Radii of thefe circles will be different: And the finding of thefe Radii is the bufinefs of this Chapter.

70. And, becaufe all curves, but circles, are formed or generated, or may be conceived to be formed or generated, by the evolution, or winding off, of fome other curves; therefore, the centres of the circles which coincide or have equal degrees of curvature with the different Points or rather Increments of the curves thus formed or generated will always be in the curves to be unwound; which curves are called the *Evolutes*; and the others formed or generated, or conceived to be formed or generated, by their evolution, are called the *Involutes*.

71. Thus, let DEF be any curve, called an evolute; round which conceive a thread to be wound and extended beyond the curve from D in a right line to A; and let this thread be evolved, or wound off, from the curve DF, so that it be continually stretched at it's full length as it leaves the curve: then will the point A generate, or describe, the involute curve ABY; and the right lines AD, BE, YF, will be the radii of curvature at the points A, B, Y, respectively. *Fig.* 44.

Corollaries.

1. The radius of curvature BE will always be equal to the length of the curve ED and right line DA: and, consequently, if the vertical distance, or shortest radius DA, vanishes; that is, if the radius at A be nothing; then, the involute curve will begin at D; and the curve DE will be equal to the radius of curvature at the point B.

2. Because (18 E. 3.) the radius of a circle is perpendicular to the tangent, the radius of curvature at any point B is always perpendicular to a tangent at that point.

3. The radius BE, which is perpendicular to the involute at the point B, is a tangent to the evolute at the point E.

PROBLEM.

72. To deduce a General Expression for BE the Radius of Curvature at any point B in the Involute Curve ABY whose Axis is AX and Evolute DE.

Put the abscis AC $= x$, and ordinate CB $= y$: and suppose bE indefinitely near to BE, bc

indefinitely near and parallel to BC; and Bm parallel to AX; that is, let Cc or B$n = x'$, and $nb = y'$. Then, Bb being confidered as a little right line coinciding with a tangent to the point B, the right angled triangles Bnb and BCH will be fimilar; (for $\angle$ EBb = $\angle$ CBn, and therefore, the $\angle$ EBn being common, the $\angle$ s nBb and CBH are equal; *ergo*, &c.) and, the $\angle$ Bbm being right, the right angled triangles mnb and bnB will, by 8 E. 6, be fimilar alfo. Wherefore, by 4 E. 6, Bn : nb :: BC : CH, that is, $x' : y'$ $:: y : \frac{yy'}{x'} =$ CH; and therefore, by 47 E. 1. BH $= \overline{BC^2 + CH^2}|^{\frac{1}{2}} = \overline{y^2 + \frac{y^2 y'^2}{x'^2}}|^{\frac{1}{2}} = \frac{y}{x'} \times \overline{x'^2 + y'^2}|^{\frac{1}{2}}$; alfo, AH $= x + \frac{yy'}{x'}$: Again, Bn : nb :: bn : nm, that is, $x' : y' :: y' : nm = \frac{y'^2}{x}$; $\because$ B$m = x' + \frac{y'^2}{x'}$. Now, becaufe the Direction of the curve ABY approaches continually nearer to a Parallelifm with the axis AX; therefore, if we fuppofe the abfcifs (AC $= x$,) to flow with an equable or uniform motion, that is, fuppofing x' or $\dot{x}$ to be invariable, or always of the fame value; then, the Increment of the ordinate (CB $= y$,) or the Velocity or Fluxion with which it flows, muft continually decreafe; that is, the *Second Moment* or *Second Fluxion* of y will be negative: and therefore, Hb, the Increment of AH viz. the Increment of $x + \frac{yy'}{x'}$, will be =

$x' + \frac{y'^2 - yy''}{x'}$. Now, the triangles EB$m$ and EHb are evidently ſimilar; therefore Bm — Hb : Bm :: (BE — HE =) BH : BE, that is, $\frac{yy''}{x'} : x' + \frac{y'^2}{x'} :: \frac{y}{x'} \times \overline{x'^2 + y'^2}\big|^{\frac{1}{2}} : \frac{\overline{x'^2 + y'^2}\big|^{\frac{3}{2}}}{x'y''}$ = BE; or (*art.* 7.) BE $= \frac{\overline{\dot{x}^2 + \dot{y}^2}\big|^{\frac{3}{2}}}{\dot{x}\ddot{y}}$.

Or, With the radius EB deſcribe the circular Arch BK; which Arch will therefore have the ſame Degree of Curvature with the Involute Curve AB at the point B. Draw the radius EK parallel to the axis AX; and produce the ordinate BC to L, to which draw AN parallel. Put the abſciſs AC = x, ordinate CB = y, radius EB or EK = r, KN = a, and NA = b; then, LE = $r—a—x$. Now, if we ſuppoſe the abſciſs, x, to increaſe uniformly, and Bm to be a tangent to the point B; then, if we draw mn parallel to BC, and Bn parallel to AX, by *art.* 24. Bn, nm, and mB, will be as the *Fluxions* of the abſciſs, ordinate, and curve, reſpectively; that is, Bn will be as $\dot{x}$, nm as $\dot{y}$, and (becauſe by 47 E. 1; Bm = $\overline{Bn^2 + nm^2}\big|^{\frac{1}{2}}$,) B$m$ as $\overline{\dot{x}^2 + \dot{y}^2}\big|^{\frac{1}{2}}$. Now, the triangles B$nm$ and BLE are ſimilar; therefore, by 4 E. 6, Bn : nm :: BL : LE, that is, $\dot{x} : \dot{y} :: y + b : r—a—x$; ∵ $r\dot{x}—a\dot{x}—x\dot{x} = y\dot{y} + b\dot{y}$; the Fluxion of which equation (ſuppoſing $\dot{x}$ invariable, and therefore, the direction of the curve AB continually approaching towards a paralleliſm with it's axis, the Fluxion of $\dot{y}$ as negative,) is $-\dot{x}^2 = \dot{y}^2 - y\ddot{y} - b\ddot{y}$; ∵ $\dot{x}^2 + \dot{y}^2 = \overline{b + y} \times \ddot{y}$, Fig. 45.

Again, by 4 E. 6, LB : BE :: nB : Bm, that is, $b+y : r :: \dot{x} : \overline{\dot{x}^2+\dot{y}^2}|^{\frac{1}{2}}$; $\because b+y = \frac{r\dot{x}}{\overline{\dot{x}^2+\dot{y}^2}|^{\frac{1}{2}}}$; which ſubſtituted for $b+y$ makes the above $\dot{x}^2 + \dot{y}^2 = \frac{r\dot{x}\ddot{y}}{\overline{\dot{x}^2+\dot{y}^2}|^{\frac{1}{2}}}$; therefore, $\overline{\dot{x}^2+\dot{y}^2} \times \overline{\dot{x}^2+\dot{y}^2}|^{\frac{1}{2}} = r\dot{x}\ddot{y}$, that is, $\overline{\dot{x}^2+\dot{y}^2}|^{\frac{3}{2}} = r\dot{x}\ddot{y}$: $\because \frac{\overline{\dot{x}^2+\dot{y}^2}|^{\frac{3}{2}}}{\dot{x}\ddot{y}} = r$ = BE; as before.

73. *Note.* When the abſciſs, x, flows with an uniform motion; it follows from *art.* 61, that the ordinate, y, flows with a retarded motion when it increaſes and the curve is concave, or when it decreaſes and the curve is convex towards the axis; and with an accelerated motion when it decreaſes and the curve is concave, or when it increaſes and the curve is convex towards the axis. Now, when y increaſes with an accelerated motion or it's *Second Fluxion* is affirmative, the General Expreſſion for the Radius of Curvature will be = $\frac{\overline{\dot{x}^2+\dot{y}^2}|^{\frac{3}{2}}}{-\dot{x}\ddot{y}}$; where the Negative ſign ſhews it's Poſition.

74. Hence, becauſe we may ſubſtitute 1 for any invariable Fluxion *; if we put $\dot{x} = 1$, the General Expreſſion for the Radius of Curvature will be = $\frac{\overline{1+\dot{y}^2}|^{\frac{3}{2}}}{\ddot{y}}$ when y increaſes with a re-

* The substituting *unity*, or 1, for an invariable Fluxion, has no other effect than it's making the operation less laborious; and, in reality, it is no more than making *unity* the Standard of the other Fluxions, or reducing the other Fluxions to a comparison with 1.

tarded motion, or it's *Second Fluxion* is Negative; and $= \frac{\overline{1+\dot{y}^2}|^{\frac{3}{2}}}{-\ddot{y}}$ when $\dot{y}$ increases with an Accelerated motion, or it's *Second Fluxion* is Affirmative. The former takes place when the curve is Concave, and the latter when it is Convex towards the axis.—Wherefore, if we put the Equation of the given curve, expressing the relation between the abscis x and ordinate y, into Fluxions, making $\dot{x} = 1$; or, from the nature of the curve, find the value of $\dot{x} = 1$ in terms of x, y, and $\dot{y}$; and then put this *fluxional* Equation into Fluxions again, still substituting 1 for $\dot{x}$, and making the Fluxion of $\dot{y}$ Negative when the curve is Concave, and Affirmative when it is Convex towards the axis; from thence the Values of the *second* and *square* of the *first Fluxions* of y may be determined: which being substituted for them in one of these two general expressions, viz. in the former when the Fluxion of $\dot{y}$ is Negative; and in the latter when it is Affirmative; we shall have a definitive expression for BE, that is, an expression for it free from Fluxions, or, the Radius of Curvature required.

Note.

75. The Vertical Distance, or Radius AD, may be obtained, by writing for $\dot{x}$ and $\dot{y}$ their values, in the General Expression for the Subnormal CH, which was found in *art.* 72. ($= \frac{yy'}{x'}$, that is, by writing $\dot{x}$ for x', and $\dot{y}$ for y',) $= \frac{y\dot{y}}{\dot{x}}$; or, making

$\dot{x} = 1$, by ſubſtituting the value of $\dot{y}$ in $y\dot{y}$, that is, by multiplying the value of $\dot{y}$ by y; and then making y vaniſh in the definitive expreſſion which will then be found.—For, the expreſſion for the Subnormal CH being the ſame at whatever point in the curve B is taken; therefore, if it be taken at A, where y vaniſhes or becomes $= 0$, the point C muſt of conſequence coincide with the vertex A, and the points E and H with D: therefore, &c.

Example I.

Fig. 46. 76. To find the Radius of Curvature at any point B in the *Parabola* AY.

Put the parameter $= a$, abſciſs AC $= x$, and ordinate CB $= y$. Now, by a well known property of the curve, $ax = y^2$; the Fluxion of which equation is $a\dot{x} = 2y\dot{y}$, or, making $\dot{x} = 1$, it is $a = 2y\dot{y}$; therefore, $\dot{y} = \frac{a}{2y} = \frac{a}{2 \times \overline{ax}|^{\frac{1}{2}}}$, for y is $= \overline{ax}|^{\frac{1}{2}}$ by the equation of the curve: And the Fluxion of this equation again, (the direction of the curve approaching continually towards a parallelism with the axis, and therefore the Fluxion of $\dot{y}$ being negative, *art.* 61.) is $-\ddot{y} = \frac{-a^2}{4 \times \overline{ax}|^{\frac{3}{2}}}$: Whence, $\dot{y}^2 = \frac{a^2}{4 \times ax} = \frac{a}{4x}$, and $\ddot{y} = \frac{a^2}{4 \times \overline{ax}|^{\frac{3}{2}}}$. Now, if for $\dot{y}^2$ and $\ddot{y}$ we ſubſtitute theſe their values, we ſhall have $\frac{\overline{1 + \dot{y}^2}|^{\frac{3}{2}}}{\ddot{y}}$, the ge-

neral expreffion for the Radius of Curvature BE

$$(art.\ 74.) = \frac{\overline{1 + \frac{a}{4x}}\big|^{\frac{3}{2}} \times \overline{4ax}\big|^{\frac{3}{2}}}{a^2} = \frac{\overline{4ax + a^2}\big|^{\frac{3}{2}}}{2a^2}.$$

Construction.

Through the point B defcribe the femicircle ABn; bifect Cn in H; make $Hr = 2AC$; and drop the perpendicular rE, terminated by the right line BE drawn through the point H: then will BE be the Radius of Curvature at the point B. For by 35 E. 3. $BC^2 = AC \times Cn$, or $\frac{BC^2}{AC} = Cn$, that is, $\frac{ax}{x} = a = Cn$, and therefore $CH = \frac{1}{2}a$, and $Cr = \frac{1}{2}a + 2x$: by 47 E. 1. $\overline{CH^2 + CB^2}\big|^{\frac{1}{2}} = BH$, that is, $\overline{\frac{1}{4}a^2 + ax}\big|^{\frac{1}{2}} = BH$; and by 4 E. 6. $CH : HB :: Cr : BE$, that is, $\frac{1}{2}a : \overline{\frac{1}{4}a^2 + ax}\big|^{\frac{1}{2}} :: \frac{1}{2}a + 2x : \frac{a + 4x}{a} \times \overline{\frac{1}{4}a^2 + ax}\big|^{\frac{1}{2}} = \frac{\overline{a^2 + 4ax}\big|^{\frac{3}{2}}}{2a^2} = BE.$

Note.

77. By writing for $\dot{y}$ its value $\frac{a}{2y}$, in $y\dot{y}$, (*art.* 75.) we have $\frac{a}{2}$ or $\frac{1}{2}a = AD$ the vertical Diftance.

Which ſame Truth may be inferred from the expreſſion for the Radius BE; for, when the ſaid Radius becomes the vertical Diſtance, that is, when the point B coincides with A, x vaniſhes; and therefore, by ſtriking $4ax$ out of the ſaid expreſſion, we have $\frac{\overline{a^2}|^{\frac{3}{2}}}{2a^2} = \frac{1}{2}a$; as before.

Example II.

78. To find the Radius of Curvature at any point B in the *Cycloid* ABD.*

Fig. 47.

Put the radius OF or OD $= a$, abſciſs AC $= x$, ordinate CB $= y$, ſine IG $= s$, and arch FG $= z$. Now, by 35 E. 3. IG $= \overline{\text{DI} \times \text{IF}}|^{\frac{1}{2}}$, that is, $s = \overline{2ay - y^2}|^{\frac{1}{2}}$; the Fluxion of which equation is $\dot{s} = \frac{a\dot{y} - y\dot{y}}{\overline{2ay - y^2}|^{\frac{1}{2}}}$: And by the nature of the Cycloid, (*art.* 35.) arch DG $=$ GB; and therefore, arch FG $=$ GI $+$ AC. or AC $=$ arch FG $-$ GI, that is, $x = z - s =$ (by ſubſtituting for s it's above value,) $z - \overline{2ay - y^2}|^{\frac{1}{2}}$; and the Fluxion of this equation, making $\dot{x} = 1$, is $1 = \dot{z} + \frac{y\dot{y} - a\dot{y}}{\overline{2ay - y^2}|^{\frac{1}{2}}}$. But (*art.* 72.) $\dot{z} = \overline{\dot{s}^2 + \dot{y}^2}|^{\frac{1}{2}} =$ (by writing for $\dot{s}$ it's above value,) $\overline{\frac{\overline{a\dot{y} - y\dot{y}}|^2}{2ay - y^2} + \dot{y}^2}|^{\frac{1}{2}} = \frac{a\dot{y}}{\overline{2ay - y^2}|^{\frac{1}{2}}}$.

* See how this Curve may be generated, *art.* 35. *note.*

which ſubſtituted for $\dot{x}$ makes the above equation $1 = \frac{a\dot{y}.}{\overline{2ay - y^2}|^{\frac{1}{2}}} + \frac{y\dot{y} - a\dot{y}}{\overline{2ay - y^2}|^{\frac{1}{2}}}$; that is, $1 = \frac{y\dot{y}}{\overline{2ay - y^2}|^{\frac{1}{2}}}$; therefore $\dot{y} = \frac{\overline{2ay - y^2}|^{\frac{1}{2}}}{y}$; and the Fluxion of this equation (the Fluxion of $\dot{y}$ being negative,) is $-\ddot{y} =$

$$\frac{\frac{ay\dot{y} - y^2\dot{y}}{\overline{2ay - y^2}|^{\frac{1}{2}}} - \dot{y}.\overline{2ay - y^2}|^{\frac{1}{2}}}{y^2} = \frac{-a\dot{y}}{y.\overline{2ay - y^2}|^{\frac{1}{2}}} = \text{(by}$$

writing for $\dot{y}$ it's equal,) $-\frac{a}{y^2}$; that is, $\ddot{y} = \frac{a}{y^2}$. Now, by ſubſtituting $\frac{2ay - y^2}{y^2}$ for $\dot{y}^2$, and $\frac{a}{y^2}$ for $\ddot{y}$, we have, by *art.* 74. $\frac{\overline{1 + \dot{y}^2}|^{\frac{3}{2}}}{\ddot{y}} = \frac{\overline{1 + \frac{2ay - y}{y^2}}|^{\frac{3}{2}} \times y^2}{a}$

$= \frac{\overline{2ay}|^{\frac{3}{2}}}{ay} = 2.\overline{2ay}|^{\frac{1}{2}} =$ BE the Radius of Curvature required.

Conſtruction.

Make FH = GB; and through the point H draw the right line BE, making BH = HE = chord GF: then will BE be the Radius of Curvature at the point B. For *art.* 35. a tangent to the point B is parallel to the chord DG, and by *art.* 71. *corol.* 2, the Radius of Curvature is always perpendicular to the tangent; therefore,

becaufe by 31 E. 3. the $\angle$ DGF is right BE muft be parallel to the chord GF. Now, by 4 and 8 E. 6. DF : FG :: GF : FI or CB, $\therefore$ $GF = \overline{DF \times CB}|^{\frac{1}{2}} = \overline{2ay}|^{\frac{1}{2}}$, and $2GF = 2.\overline{2ay}|^{\frac{1}{2}} = BE$.

Note.

79. By *art.* 75. if we multiply the value of $\dot{y}$, viz. $\frac{\overline{2ay - y^2}|^{\frac{1}{2}}}{y}$, by y; we fhall have the Subnormal $CH = \overline{2ay - y^2}|^{\frac{1}{2}}$; which, when y vanifhes, becomes $= 0$, and equal to the vertical Diftance: fo that the Vertices of the Evolute and Involute Curves coincide.

EXAMPLE III.

Fig. 48. 80. To find the Radius of Curvature at any point B in the Curve AD; whofe nature is fuch, that the Triangle CBT, made of the Ordinate, Tangent, and Subtangent, is always proportional to the Ordinate CB; or, whofe Subtangent CT is equal to a given line $= a$. (See *art.* 36.)

Put $GC = x$, and $CB = y$; then, by *art.* 25. $\frac{\dot{x}y}{\dot{y}} = a$, that is, if $\dot{x}$ be made $= 1$, $\frac{y}{\dot{y}} = a$; $\therefore \dot{y} = \frac{y}{a}$; therefore, $\dot{y}^2 = \frac{y^2}{a^2}$, and (becaufe here y flows with an accelerated motion, or it's Second

Fluxion is affirmative,) $\ddot{y} = \frac{\dot{y}}{a}$, that is, (by ſubſtituting $\frac{y}{a}$ for $\dot{y}$ it's value,) $\ddot{y} = \frac{y}{a^2}$. Now, by writing for $\dot{y}^2$ and $\ddot{y}$ theſe their values, in $\frac{\overline{1+\dot{y}^2}|^{\frac{3}{2}}}{-\ddot{y}}$, (*art.* 74.) we have $\frac{\overline{1+\frac{y^2}{a^2}}|^{\frac{3}{2}} \times a^2}{-y} = \frac{\overline{a^2+y^2}|^{\frac{3}{2}}}{-ay} =$ the Radius of Curvature ſought: Where the Negative ſign only ſhews, that the Evolute and Radius of Curvature lie on the other ſide of the curve with regard to x and y.

Conſtruction.

Draw Bn parallel to CT, Tn perpendicular to TB, nE perpendicular to nB, and BE parallel to Tn: then will BE be the Radius of Curvature at the point B. For, by 47 E. 1. BT $= \overline{TC^2+CB^2}|^{\frac{1}{2}} = \overline{a^2+y^2}|^{\frac{1}{2}}$, and by 4 E. 6. CT : TB :: TB : Bn, that is, $a : \overline{a^2+y^2}|^{\frac{1}{2}} :: \overline{a^2+y^2}|^{\frac{1}{2}} : \frac{a^2+y^2}{a} =$ Bn; and, CB : BT :: nB : BE, that is, $y : \overline{a^2+y^2}|^{\frac{1}{2}} :: \frac{a^2+y^2}{a} : \frac{\overline{a^2+y^2}|^{\frac{3}{2}}}{ay} =$ BE.

EXAMPLE IV.

Fig. 49. 81. To find the Radius of Curvature at any point B in the Curve AD; whose nature is such, that the Tangent BT is every-where equal to a given line $= a$.

Put GC $= x$, and CB $= y$; then, by 47 E. 1, $\overline{TB^2 - BC^2}|^{\frac{1}{2}} = CT$, that is, $\overline{a^2 - y^2}|^{\frac{1}{2}} = CT =$ (*art.* 25.) $\frac{\dot{x}y}{\dot{y}}$; or, making $\dot{x} = 1$, $\overline{a^2 - y^2}|^{\frac{1}{2}} = \frac{y}{\dot{y}}$; $\because$ $\dot{y} = \frac{y}{\overline{a^2 - y^2}|^{\frac{1}{2}}}$; therefore $\dot{y}^2 = \frac{y^2}{a^2 - y^2}$, and (because the Fluxion of $\dot{y}$ is affirmative,) $\ddot{y} = \frac{\dot{y} \times \overline{a^2 - y^2}|^{\frac{1}{2}} + \frac{y^2\dot{y}}{\overline{a^2 - y^2}|^{\frac{1}{2}}}}{a^2 - y^2}$, that is, by substituting for $\dot{y}$ it's value, $\ddot{y} = \frac{y + \frac{y^3}{a^2 - y^2}}{a^2 - y^2} = \frac{a^2 y}{\overline{a^2 - y^2}|^2}$. Now by writing for $\dot{y}^2$ and $\ddot{y}$ these their values, in $\frac{\overline{1 + \dot{y}^2}|^{\frac{3}{2}}}{-\ddot{y}}$, (*art.* 74.) we have $\frac{\overline{1 + \frac{y^2}{a^2 - y^2}}|^{\frac{3}{2}} \times \overline{a^2 - y^2}|^2}{-a^2 y}$ $= \frac{\overline{a^2}|^{\frac{3}{2}} \times \overline{a^2 - y^2}|^{\frac{1}{2}}}{-a^2 y} = -\frac{a}{y} \times \overline{a^2 - y^2}|^{\frac{1}{2}} =$ the Radius of Curvature required: Where the Negative sign shews it's position.

Construction.

On the extremity of the ſubtangent, T, erect the perpendicular TE; and draw the right line BE perpendicular to the tangent TB: then will BE be the Radius of Curvature at the point B; or, the point E will be in the Evolute Curve. For, the triangles CBT and BTE will be ſimilar; and therefore, by 4 E. 6. BC : CT :: TB : BE,

that is, $y : \overline{a^2 - y^2}|^{\frac{1}{2}} :: a : \frac{a}{y} \times \overline{a^2 - y^2}|^{\frac{1}{2}} = \text{BE}.$

82. The General Expreſſion for the Radius of Curvature found in *art.* 74. being only for Curves referred to an Axis; we ſhall now deduce one for *Spira's*, or thoſe Curves whoſe ordinates are referred to a fixed or central Point.

83. Let CBY be the Curve; C the central point, or that from which all the ordinates iſſue; and BE the Radius of Curvature at the point B, that is, let the point E be ſuppoſed in the Evolute *Fig.*
Curve: conceive C*b* and E*b* indefinitely near to 50.
CB and EB, that is, let the points B and *b* be ſuppoſed indefinitely near to each other; and let CF and C*f* be perpendicular to EB and E*b* reſpectively: then will the points F and *r* be indefinitely near to a coincidence; and therefore, *art.* 7. B*r* and C*r* may be taken as equal to BF and CF. Now, if with the ordinate CB, as a radius, the little circular arch B*n* be deſcribed and conſidered as a little right line perpendicular to C*b*; and the increment B*b* be conſidered as coinciding with a tangent to the point B; then, the little right-angled triangle B*nb* will be ſimilar to the right-angled triangle BFC; (for ∠ CB*n* = ∠ EB*b*;

and therefore, ∠ EBn being common, the ∠ CBF = ∠ nBb; and consequently, the angles at F and n being right, ∠ BCF = ∠ Bbn;) therefore, by 4 E. 6. bB : Bn :: CB : BF; that is, (if we put the ordinate CB = y, Bn = x', and nb = y', when by 47 E. 1. Bb will be = $\overline{x' + y'^2}|^{\frac{1}{2}}$,) $\overline{x'^2+y'^2}|^{\frac{1}{2}} : x'$ :: y : $\dfrac{xy}{\overline{x'^2+y'^2}|^{\frac{1}{2}}}$ = BF or Br; and Bb : bn :: BC : CF, that is, $\overline{x'^2+y'^2}|^{\frac{1}{2}} : y' :: y : \dfrac{y'}{\overline{x'^2+y^2}|^{\frac{1}{2}}}$ = CF or Cr; the Increment of which is rf; that is, (supposing x' to be invariable)

$$\frac{\overline{y'^2 + yy''} \times \overline{x'^2 + y'^2}|^{\frac{1}{2}} - \dfrac{y'y'' \times yy'}{\overline{x'^2+y'^2}|^{\frac{1}{2}}}}{x'^2+y'^2} =$$

$\dfrac{x'^2y'^2+y'^4+yx'^2y''}{\overline{x'^2+y'^2}|^{\frac{3}{2}}} = rf$. Again, the triangles EBb and Erf being similar, Bb — rf : Bb :: (BE — rE, or) rB : BE; that is, $(\overline{x'^2+y'^2}|^{\frac{1}{2}}$ $- \dfrac{x'^2y'^2+y'^4+yx'^2y''}{\overline{x'^2+y'^2}|^{\frac{3}{2}}} =) \dfrac{x'^4 + x'^2y'^2 - yx'^2y''}{\overline{x'^2+y'^2}|^{\frac{3}{2}}}$: $\overline{x'^2+y'^2}|^{\frac{1}{2}} :: \dfrac{x'y}{\overline{x'^2+y'^2}|^{\frac{1}{2}}} : \dfrac{y \times \overline{x'^2+y'^2}|^{\frac{3}{2}}}{x'^3+x'y'^2-yx'y''}$ = BE;

or, *art.* 7. BE $= \dfrac{y \times \overline{\dot{x}^2+\dot{y}^2}|^{\frac{3}{2}}}{\dot{x}^3+\dot{x}\dot{y}^2-y\dot{x}\ddot{y}}$; which is a General Expression for the Radius of Curvature of all Curves referred to a fixed or central Point, when x' or $\dot{x}$ is invariable.

84. Hence, if $\dot{x}$ be made $= 1$, the General Expreſſion for the Radius of Curvature will be $=$ $\frac{y \times \overline{1+\dot{y}^2}|^{\frac{3}{2}}}{1+\dot{y}^2-y\ddot{y}}$.—Wherefore, if we put the Equation of the given Spiral into Fluxions, (making $\dot{x} = 1$,) and put this *fluxional* Equation into Fluxions again; and from thence, or from the nature of the curve, find the values of $\dot{y}^2$ and $\ddot{y}$: then, if for $\dot{y}^2$ and $\ddot{y}$ we ſubſtitute theſe their values, in this General Expreſſion, we ſhall have BE the Radius of Curvature required: As in the following Examples.

Example I.

85. To find the Radius of Curvature at any point B in the *Spiral* of *Archimedes*, CB, &c*. *Fig.* 51.

Put the circumference of the generating circle AF, &c. $= a$, and it's radius CA $= b$; ordinate CB $= y$, arch AF $= z$. Let Cf be ſuppoſed indefinitely near to CF, that is, let the $\angle$ FCf be ſuppoſed indefinitely ſmall; and with the ordinate CB as a radius, deſcribe the little circular arch Bn, which put $= x'$; alſo, put Ff $= z'$. Now, by the nature of the curve, $a : b :: z : y$, or $z = \frac{ay}{b}$, the Fluxion of which equation is $\dot{z} = \frac{a\dot{y}}{b}$: and, by the ſimilar ſectors CBn and CFf, $y : x' :: b : z' =$

* See how this Curve is generated, *art.* 39. *note.*

$\frac{bx'}{y}$, or, *art.* 7. $\dot{z} = \frac{b\dot{x}}{y}$. Hence $\frac{a\dot{y}}{b} = \frac{b\dot{x}}{y}$; that is, (making $\dot{x} = 1$,) $\frac{a\dot{y}}{b} = \frac{b}{y}$; from which equation we have $\dot{y} = \frac{b^2}{ay}$; therefore $\dot{y}^2 = \frac{b^4}{a^2y^2}$, and $\ddot{y} = \frac{-ab^2\dot{y}}{a^2y^2} =$ (by writing for $\dot{y}$ it's value,) $\frac{-b^4}{a^2y^3}$: And, if we substitute for $\dot{y}^2$ and $\ddot{y}$ these their values, we shall have $\frac{y \times \overline{1+\dot{y}^2}|^{\frac{3}{2}}}{1+\dot{y}^2-y\ddot{y}}$ (*art.* 84.) $=$

$$\frac{y \times \overline{1+\frac{b^4}{a^2y^2}}\Big|^{\frac{3}{2}}}{1+\frac{b^4}{a^2y^2}+\frac{b^4}{a^2y^2}} = \frac{\overline{a^2y^2+b^4}|^{\frac{3}{2}}}{a^3y^2+2ab^4}$$ = BE, the Radius of Curvature sought.

Construction.

Through the center C draw the indefinite right line Hv perpendicular to the ordinate CB; draw the tangent TB, perpendicular to which draw BH; produce BC to V, making BR = TH, and RV = CH: with BV and BR, as radii, describe the arches Vv and Rr, draw the right line vB; and from the intersecting point r draw rE parallel to vH: then will BE be the Radius of Curvature at the point B. For, (*art.* 39.) CT = $\frac{yz}{v}$, that is, by substituting $\frac{ay}{b}$ for z, CT = $\frac{ay^2}{b^2}$;

and by 3 and 4 E. 6. TC : CB :: CB : CH, that is, $\frac{ay^2}{b^2} : y :: y : \frac{b^2}{a} = $ CH; therefore TH $=$ BR $=$ Br $= \frac{ay^2}{b^2} + \frac{b^2}{a}$, and BV $=$ Bv $= \frac{ay^2}{b^2} + \frac{2b^2}{a}$ and by 47 E. 1. HB $= \overline{BC^2 + CH^2}|^{\frac{1}{2}} = \overline{y^2 + \frac{b^4}{a^2}}|^{\frac{1}{2}} = \frac{\overline{a^2y^2 + b^4}|^{\frac{1}{2}}}{a}$: Again, by 4 E. 6. B$v$: BH :: Br : BE, that is, $\frac{ay^2}{b^2} + \frac{2b^2}{a} : \frac{\overline{a^2y^2 + b^4}|^{\frac{1}{2}}}{a} :: \frac{ay^2}{b^2} + \frac{b^2}{a} : \frac{\overline{a^2y^2 + b^4}|^{\frac{1}{2}}}{a} \times \frac{a^2y^2 + b^4}{a^2y^2 + 2b^4} = \frac{\overline{a^2y^2 + b^4}|^{\frac{3}{2}}}{a^3y^2 + 2ab^4} = $ BE.

Example II.

86. To find the Radius of Curvature at any point B in the *Logarithmic Spiral* CBY; whose Equation (putting the ordinate CB $= y$, curve CB $= z$, and a and b for two given quantities,) is $az = by$. (See *art.* 40.) *Fig.* 52

The Fluxion of the equation of the curve is $a\dot{z} = b\dot{y}$; therefore $\dot{z} = \frac{b\dot{y}}{a}$. Let the angle BC$b$ be supposed indefinitely small; and with the ordinate CB, as a radius, let the little circular arch Bn be described. Now, if we consider Bn as a little right

line perpendicular to Cb, and Bb as a little right line coinciding with a tangent to the point B; then, by 47 E. 1. $Bb = \overline{Bn^2 + nb^2}|^{\frac{1}{2}}$, that is, (putting $Bn = x'$, $nb = y'$, and $Bb = z'$,) $z' = \overline{x'^2 + y'^2}|^{\frac{1}{2}}$, or, by substituting the Fluxion for the Increment, $\dot{z} = \overline{\dot{x}^2 + \dot{y}^2}|^{\frac{1}{2}}$, that is, (if we put $\dot{x} = 1$,) $\dot{z} = \overline{1 + \dot{y}^2}|^{\frac{1}{2}}$. Hence $\frac{b\dot{y}}{a} = \overline{1 + \dot{y}^2}|^{\frac{1}{2}}$; which equation squared is $\frac{b^2\dot{y}^2}{a^2} = 1 + \dot{y}^2$; and this produces $\dot{y}^2 = \frac{a^2}{b^2 - a^2}$; therefore, $\dot{y} = \frac{a}{\overline{b^2 - a^2}|^{\frac{1}{2}}}$; and, this being an invariable quantity, therefore $\ddot{y} = 0$. Now, by writing for $\dot{y}^2$ and $\ddot{y}$ these their values, the general expression for the Radius of Curvature, viz. $\frac{y \times \overline{1 + \dot{y}^2}|^{\frac{3}{2}}}{1 + \dot{y}^2 - y\ddot{y}}$ *art.* 84. will become

$$= \frac{y \times \overline{1 + \frac{a^2}{b^2 - a^2}}\Big|^{\frac{3}{2}}}{1 + \frac{a^2}{b^2 - a^2} - 0} = y \times \overline{\frac{b^2}{b^2 - a^2}}\Big|^{\frac{1}{2}} = y \times \frac{b}{\overline{b^2 - a^2}|^{\frac{1}{2}}} = \text{BE},$$

the Radius of Curvature required.

Construction.

Draw the tangent TB; perpendicular to which draw the right line BE, terminated by the sub-

tangent TC produced: then will the point E be in the Evolute curve; or, the right line BE will be the Radius of Curvature at the point B. For then (TE being perpendicular to the ordinate CB,) by 8 E. 6. the triangles CTB and CBE will be similar; and therefore, by 4 E. 6. CT : TB :: CB : BE; that is, (becauſe by *art.* 40. CT : TB :: $\overline{b^2 - a^2}|^{\frac{1}{2}} : b$,) $\overline{b^2 - a^2}|^{\frac{1}{2}} : b :: y : y \times \dfrac{b}{\overline{b^2 - a^2}|^{\frac{1}{2}}} = \text{BE}$.

CHAPTER VII.

Of finding the Nature of the Evolute of a given Involute Curve.

As it is abſolutely neceſſary for the Learner to be well acquainted with the foregoing Chapter before he enters upon this, we ſhall not here define the meaning of Evolute and Involute curves, it being ſufficiently explained therein.

87. Let BE be the Radius of Evolution (or Curvature) at any point B in the Involute curve AB, whoſe abſciſs is AC $= x$, and ordinate CB $=$ y. Parallel to HA draw EN; produce BC to L; and, equal and parallel to CL, draw DN from the vertex of the Evolute DE. Then will the triangles BHC and BEL be ſimilar; and therefore, by *Fig.* 53.

4 E. 6. BH : HC :: BE : EL, that is, (by *art.* 72 and 7.) $\frac{y}{\dot{x}} \times \overline{\dot{x}^2 + \dot{y}^2}|^{\frac{1}{2}} : \frac{y\dot{y}}{\dot{x}} :: \frac{\overline{\dot{x}^2 + \dot{y}^2}|^{\frac{3}{2}}}{\dot{x}\ddot{y}} : \dot{y} \times \frac{\dot{x}^2 + \dot{y}^2}{\dot{x}\ddot{y}} = EL$; and HC : CB :: EL : LB, that is, $\frac{y\dot{y}}{\dot{x}} : y :: \dot{y} \times \frac{\dot{x}^2 + \dot{y}^2}{\dot{x}\ddot{y}} : \frac{\dot{x}^2 + \dot{y}^2}{\ddot{y}} = LB$. Now, these are General Expressions for EL and LB, when $\dot{x}$ is considered as invariable, and the Fluxion of $\dot{y}$ as negative. Hence therefore,

88. If $\dot{x} = 1$, and the Fluxion of $\dot{y}$ be negative; the General Expression for BL will be $= \frac{1 + \dot{y}^2}{\ddot{y}}$, and this multiplied by $\dot{y}$ is $\dot{y} \times \frac{1 + \dot{y}^2}{\ddot{y}} =$ the General Expression for LE. Now, by help of the Equation of the given Involute curve, exterminate $\dot{y}$, $\dot{y}^2$, and $\ddot{y}$, out of these expressions, as in the preceding Chapter; and, by *art.* 75. find the vertical distance AD. Then, if we put the abscis of the Evolute DN $= u$, and it's ordinate NE $= v$; by help of these two equations, $u =$ BL — BC, and $v =$ AC — AD + LE, we may get the Nature of the Evolute curve DE required.

89. *Note.* If the given Involute be convex towards it's axis, and, x and y increase together, or the Fluxions of x and $\dot{y}$ be both affirmative; then, the General Expressions for BL and LE will be $\frac{1 + \dot{y}^2}{-\ddot{y}}$ and $\dot{y} \times \frac{1 + \dot{y}^2}{-\ddot{y}}$ respectively; wherein, the Negative sign shews, that, the points L

and E muſt be taken on the Concave ſide of the Involute curve, that is, on the other ſide of it with regard to x and y.

EXAMPLE I.

90. To find the Nature of the Curve DE, by whoſe Evolution the *Parabola* AB is deſcribed.

Put AC $= x$, CB $= y$; and DN $= u$, NE $= v$. Now, (by *art.* 76.) $\dot{y} = \frac{a}{2 \times \overline{ax}|^{\frac{1}{2}}}$, $\dot{y}^2 = \frac{a}{4x}$, and $\ddot{y} = \frac{a^2}{4 \times \overline{ax}|^{\frac{3}{2}}}$; which values of $\dot{y}$, $\dot{y}^2$, and $\ddot{y}$, being ſubſtituted for them, make the General Expreſſion for BL, *viz.* $\frac{1 + \dot{y}^2}{\ddot{y}}$ (*art.* 88.) become

$$= \frac{\overline{1 + \frac{a}{4x}} \times 4 . \overline{ax}|^{\frac{3}{2}}}{a^2} = \frac{\overline{4x + a} \times \overline{ax}|^{\frac{1}{2}}}{a};$$ and

that for LE, *viz.* $\dot{y} \times \frac{1 + \dot{y}^2}{\ddot{y}} = \dot{y} \times \text{BL} =$

$$\frac{a}{2 \times \overline{ax}|^{\frac{1}{2}}} \times \frac{\overline{4x + a} \times \overline{ax}|^{\frac{1}{2}}}{a} = 2x + \tfrac{1}{2}a.$$ Hence,

$$u\ (= \text{BL} - \text{BC}) = \frac{\overline{4x + } \times \overline{ax}|^{\frac{1}{2}}}{a} - (y \text{ or}) \overline{ax}|^{\frac{1}{2}}$$

$= \overline{\frac{4x.ax}{a}}\Big|^{\frac{1}{2}}$; and (becauſe by *art.* 77. the vertical diſtance AD is $= \frac{1}{2}a$,) v ($=$ AC — AD + LE) $= x - \frac{1}{2}a + 2x + \frac{1}{2}a = 3x$. Now, the former of theſe two equations produces $x^3 = \frac{au^2}{16}$; and, the cube of the latter, divided by 27, is $x^3 = \frac{v^3}{27}$: therefore, $\frac{au^2}{16} = \frac{v^3}{27}$, and $\frac{27a}{16}u^2 = v^3$; which is the equation of the Curve DE, expreſſing the relation between the abſciſs and ordinate; and, the Equation of the Semicubical Parabola (whoſe parameter is $\frac{27a}{16}$,) being the ſame; therefore, the Evolute DE is a *Semicubical Parabola*, whoſe vertex is D.

Example II.

Fig. 54. 91. To find the Nature of the Curve AEP, by whoſe Evolution the *Cycloid* ABD is deſcribed.*

Put AC$=x$, CB$=y$, arch FG$=z$, and OD or OF $= a$; then (*art.* 78.) $\dot{y} = \overline{\frac{2ay - y^2}{y}}\Big|^{\frac{1}{2}}\dot{x}$, $\dot{y}^2 = \frac{2ay - y^2}{y^2}$,

* See in what manner a *Cycloid* is generated, *art.* 35, *note.*

and $\ddot{y} = \frac{a}{y^2}$: wherefore, (*art.* 88.) BL $= \frac{1 + \dot{y}^2}{\ddot{y}} =$

$\frac{1 + \frac{2ay - y^2}{y^2} \times y^2}{a} = 2y$, and LE $= \dot{y} \times$ BL $=$

$\frac{\overline{2ay - y^2}|^{\frac{1}{2}}}{y} \times 2y = 2.\overline{2ay - y^2}|^{\frac{1}{2}}$. Hence, if we put the abscifs AN $= u$, and ordinate NE $= v$, we have u ($=$ BL $-$ CB $=$) $2y - y = y$, and v $=$ (AC $+$ LE $=$) $x + 2.\overline{2ay - y^2}|^{\frac{1}{2}}$, that is, (becaufe, *art.* 78. $x = z - \overline{2ay - y^2}|^{\frac{1}{2}}$,) $v = z$ $+ \overline{2ay - y^2}|^{\frac{1}{2}}$, or, (writing u for y it's equal,) v $= z + \overline{2au - u^2}|^{\frac{1}{2}}$. Wherefore, the Evolute curve AEP is a Cycloid, and equal to the given Cycloid ABD. For, let AS $=$ SV $= a$, then (AN being $=$ FI,) AT $=$ FG $= z$, and NT $=$ $\overline{2au - u^2}|^{\frac{1}{2}} =$ IG; and therefore AT $+$ TN $= z$ $+ \overline{2au - u^2}|^{\frac{1}{2}}$, that is, AT $+$ TN $=$ NE; which is the property of the Cycloid: therefore, the Evolute AEP is a Cycloid; and, becaufe AV $=$ FD, therefore the Cycloids AEP and ABD are equal.

EXAMPLE III.

92. To find the Nature of the Evolute of the Curve AD, whofe Tangent BT is every-where equal to a given line $= a$. *Fig.* 55.

Let BE be the Radius of Curvature at the point B; then (*art.* 81.) if a perpendicular be erected

on the point T, it will paſs through the point E; wherefore, when the point T coincides with F, that is, when the tangent and ordinate become equal or the points B and D coincide, the point E will likewiſe coincide with D: conſequently, the Vertex of the Evolute coincides with that of the Involute.

Put GF $= b$, GC $= x$, CB $= y$, DN $= u$, and NE $= v$; then, (*art.* 81.) $\dot{y} = \frac{y}{\overline{a^2 - y^2}|^{\frac{1}{2}}}$, $\dot{y}^2 = \frac{y^2}{a^2 - y^2}$, and $\ddot{y} = \frac{a^2 y}{\overline{a^2 - y^2}|^{2}}$: wherefore, (*art.* 89.) BL $= \frac{1 + \dot{y}^2}{-\ddot{y}} = \overline{1 + \frac{y^2}{a^2 - y^2}} \times - \frac{\overline{a^2 - y^2}|^2}{a^2 y}$ $= \frac{a^2 - y^2}{-y}$, and LE $= \dot{y} \times$ BL $= \frac{y}{\overline{a^2 - y^2}|^{\frac{1}{2}}} \times \frac{a^2 - y^2}{-y} = - \overline{a^2 - y^2}|^{\frac{1}{2}}$; that is, (becauſe the Negative ſign only ſhews that the points L and E muſt be taken on the Concave ſide of the Involute curve DA,) BL $= \frac{a^2 - y^2}{y}$, and LE $= \overline{a^2 - y^2}|^{\frac{1}{2}}$. Hence we have $u =$ (LB + BC − DF =) $\frac{a^2 - y^2}{y} + y - a$, which equation gives $y = \frac{a^2}{u + a}$, and therefore $\dot{y} = - \frac{a^2 \dot{u}}{\overline{u + a}|^2}$; alſo, $v =$ (GF − GC + CT =) $b - x + \overline{a^2 - y^2}|^{\frac{1}{2}}$, and there-

fore $\dot{v} = -\dot{x} - \frac{y\dot{y}}{\overline{a^2 - y^2}|^{\frac{1}{2}}} =$ (becaufe by *art.* 81.

CT $= \frac{\dot{x}y}{\dot{y}} = \overline{a^2 - y^2}|^{\frac{1}{2}}$, or, $\dot{x} = \frac{\dot{y}}{y} \times \overline{a^2 - y^2}|^{\frac{1}{2}}$,)

$-\frac{\dot{y}}{y} \times \overline{a^2 - y^2}|^{\frac{1}{2}} - \frac{y\dot{y}}{\overline{a^2 - y^2}|^{\frac{1}{2}}} = \frac{-a^2\dot{y}}{y \times \overline{a^2 - y^2}|^{\frac{1}{2}}}$,

that is, (by writing for y and $\dot{y}$ their above values affected with u and $\dot{u}$,) $\dot{v} = \frac{a\dot{u}}{\overline{u^2 + 2au}|^{\frac{1}{2}}}$; which is an Equation for the Evolute DE, and is alfo an Equation of the *Catenary* curve : therefore, the Evolute DE is the *Catenary*.*

Or, The above Equation of the Evolute may be found thus.—Let E$n = u'$, and $ne = v'$; then, the triangles *en*E and TBE being fimilar, we have, by 4 E. 6. $en : n$E :: TB : BE, that is, $v' : u' ::$ $a : \frac{au'}{v'} =$ BE, or (*art.* 7.) $\frac{a\dot{u}}{\dot{v}} =$ BE; but, by 47 E. 1. BE $= \overline{\text{ET}^2 - \text{TB}^2}|^{\frac{1}{2}}$, that is, (becaufe ET $=$ NF $= u + a$,) BE $= \overline{\overline{u + a}|^2 - a^2}|^{\frac{1}{2}} =$ $\overline{u^2 + 2au}|^{\frac{1}{2}}$: therefore, $\frac{a\dot{u}}{\dot{v}} = \overline{u^2 + 2au}|^{\frac{1}{2}}$, and $\dot{v} =$ $\frac{a\dot{u}}{\overline{u^2 + 2au}|^{\frac{1}{2}}}$; as before.

SCHO.

* The *Catenary* is a Curve, as ADB or *adb*, formed by a flexible line or chain hanging freely from two points of fufpension, A and B, or *a* and *b*, whether the said points be horizontal or not. *Fig.* 56.

Scholium.

93. The Evolute of a *Spiral*, or indeed of any other Curve, may be defcribed, by finding the Radii of Curvature at feveral points in the Involute: for then we fhall have as many points in the Evolute; through which, if a Curve line be drawn, it will be the Evolute fought.

PART II.

CHAP. I.

*Of Infinite Series**.

AS the learner may, perhaps, be unacquainted with *Infinite Series*, the knowledge of which is ſometimes abolutely neceſſary, in order to find the *Fluents* (or flowing quantities) of *Fluxions* expreſſed in a Fractional manner, and of ſuch wherein there are Surds or Radical quantities; and, becauſe in ſome of the following pages the *Fluents* of ſuch *Fluxional* expreſſions are to be found; the adding of this Chapter may therefore not be improper, though it is, in ſome meaſure, foreign to the buſineſs in hand.

PROB.

* The Methods of reducing compound expressions into *Infinite Series*, by division and extracting of roots, as taught in this Chapter, were invented by the great Inventor of *Fluxions* about the year 1664; who at the same time, or rather a little before, invented the celebrated *Binomial Theorem*.

PROBLEM. I.

To reduce a compound Fractional expreſſion into an *Infinite Series*; that is, into a number of terms, which, if infinitely continued, ſhall be equal to the given fractional expreſſion.

EXAMPLE I.

94. To reduce $\frac{b}{a+x}$ into an *Infinite Series.*

Place the denominator $a+x$ as a diviſor, and the numerator b as a dividend; and divide, as in common algebraic diviſion, until you have 4, 5, 6, or more terms in the Quotient; after which you may find as many terms as you pleaſe, by only conſidering the *law* of the progreſſion of the terms already found. Thus, the four firſt terms being $\frac{b}{a} - \frac{bx}{a^2} + \frac{bx^2}{a^3} - \frac{bx^3}{a^4}$, (ſee the operation below,) the *law* of the continuation of the diviſion, or of the *Series*, is plain; for each Succeeding term is evidently produced by multiplying the Preceding by $-\frac{x}{a}$; and conſequently, the fifth term will be $+\frac{bx^4}{a^5}$, the ſixth term $-\frac{bx^5}{a^6}$, &c.

Operation.

$$a + x)\,b \ldots \left(\frac{b}{a} - \frac{bx}{a^2} + \frac{bx^2}{a^3} - \frac{bx^3}{a^4} + \&c.\right.$$

$$\begin{array}{l}
b + \frac{bx}{a} \\
\hline
0 - \frac{bx}{a} \\
\quad - \frac{bx}{a} - \frac{bx^2}{a^2} \\
\hline
\qquad 0 + \frac{bx^2}{a^2} \\
\qquad\quad \frac{bx^2}{a^2} + \frac{bx^3}{a^3} \\
\hline
\qquad\qquad 0 - \frac{bx^3}{a^3} \\
\qquad\qquad - \frac{bx^3}{a^3} - \frac{bx^4}{a^4} \\
\hline
\qquad\qquad\quad 0 + \frac{bx^4}{a^4} \\
\qquad\qquad\quad \&c.
\end{array}$$

Or, If we put x before a, in the denominator of the above fractional expreſſion; that is, if the diviſor be placed thus, $x + a$, inſtead of $a + x$; then the Quotient, or *Series*, will be $\frac{b}{x} - \frac{ba}{x^2} + \frac{ba^2}{x^3} - \frac{ba^3}{x^4} + \&c.$ Whence, the *law* of the

continuation of the *Series* may be obſerved as before.

SCHOLIUM.

95. In General, in order to have a *true* or *converging* Series, or that in which the terms continually Decreaſe, the Greateſt term muſt be placed firſt. Thus, in the above Example, if a be greater than x; then a muſt be the firſt term in the diviſor, and $\frac{b}{a} - \frac{bx}{a^2} + \frac{bx^2}{a^3} - \frac{bx^3}{a^4} +$ &c. will be the *true* Series: But if x be greater than a; then x must be the firſt term in the diviſor, and $\frac{7}{x} - \frac{ba}{x^2} + \frac{ba^2}{x^3} - \frac{ba^3}{x^4} +$ &c. will be the *true* Series; the other, then, being a *diverging* one, the terms in it continually Increaſing; and conſequently the farther you go in the *Series* the farther it will be from the truth.

Though it is impoſſible to take any number of terms in the *Series* that ſhall *truly* expreſs the value of the quantity given; yet, in general, a few of the leading terms will be near enough the truth for any purpoſe.

EXAMPLE II.

96. To reduce $\frac{a^2}{a^2 + 2ax + x^2}$ into an *Infinite Series.*

Operation.

$$a^2 + 2ax + x^2) \, a^2 \ldots (1 - \frac{2x}{a} + \frac{3x^2}{a^2} - \frac{4x^3}{a^3} + \&c.$$

$$\begin{aligned}
&a^2 + 2ax + x^2 \\
\hline
&0 - 2ax - x^2 \\
&- 2ax - 4x^2 - \frac{2x^3}{a} \\
\hline
&0 + 3x^2 + \frac{2x^3}{a} \\
&3x^2 + \frac{6x^3}{a} + \frac{3x^4}{a^2} \\
\hline
&0 - \frac{4x^3}{a} - \frac{3x^4}{a^2} \\
&- \frac{4x^3}{a} - \frac{8x^4}{a^2} - \frac{4x^5}{a} \\
\hline
&0 + \frac{5x^4}{a^2} + \frac{4x^5}{a^3}
\end{aligned}$$

Now, from theſe four terms of the ſeries it is eaſy to ſee the *law* of the continuation is ſuch, that, the numerators are the powers of *x*, whoſe Indices are 1 leſs than the numbers of the terms to which they reſpectively belong, multiplied by the ſaid numbers; that, the denominators are the powers of *a*, whoſe Indices are the ſame with thoſe of the numerators; and that, the ſigns of the terms

are alternately changed. So that, the 5th term is $+\frac{5x^4}{a^4}$, the 6th term is $-\frac{6x^5}{a^5}$; and so on.

PROBLEM. II.

To reduce a compound Surd quantity into an *Infinite Series*; that is, to free a compound expression from Surds by throwing it into a number of decreasing terms, which if infinitely continued, shall be equal to the quantity given.

Example I.

97. To reduce $\overline{a^2+4y^2}|^{\frac{1}{2}}$ into an *Infinite Series.*

Take the square root of a^2, which is a, for the First term of the *Series*; (see the Operation below;) then, this squared and subtracted from a^2+4y^2, leaves $4y^2$; and this remainder divided by the double of the first term, (as in the common arithmetical extraction of the square root,) *viz.* by $2a$, gives $+\frac{2y^2}{a}$ for the Second term of the *Series*; which, with the double of the first term, being multiplied by $\frac{2y^2}{a}$ the said second term, gives $4y^2 + \frac{4y^4}{a^2}$, and this subtracted from $4y^2$ leaves $-\frac{4y^4}{a^2}$, which divided by the double of the two

firſt terms of the Series, *viz.* by $2a + \frac{4y^2}{a}$, gives $-\frac{2y^4}{a^3}$ for the Third term of the *Series*; which, with the double of the two first terms, *viz.* $2a + \frac{4y^2}{a}$ being multiplied by $-\frac{2y^4}{a^3}$ the ſaid third term, gives $-\frac{4y^4}{a^2} - \frac{8y^6}{a^4} + \frac{4y^8}{a^6}$, and this ſubtracted from $-\frac{4y^4}{a^2}$ leaves $\frac{8y^6}{a^4} - \frac{4y^8}{a^6}$, which divided by the double of the three firſt terms of the Series already found, *viz.* by $2a + \frac{4y^2}{a} - \frac{4y^4}{a^3}$, gives $+\frac{4y^6}{a^5}$ for the Fourth term of the *Series*.——After the ſame manner may be found any number of terms in the Series: And, when the *law* of the progreſſion, or of the continuation of the Series, is diſcovered, the terms may be continued on at pleaſure.

Operation.

$$a^2 + 4y^2 \left(a + \frac{2y^2}{a} - \frac{2y^4}{a^3} + \frac{4y^6}{a^5} - \&c. \right.$$

$$a^2$$

$$2a)\; 0 + 4y^2$$

$$4y^2 + \frac{4y^4}{a^2}$$

$$2a + \frac{4y^2}{a})\; 0 - \frac{4y^4}{a^2}$$

$$- \frac{4y^4}{a^2} - \frac{8y^6}{a^4} + \frac{4y^8}{a^6}$$

$$2a + \frac{4y^2}{a} - \frac{4y^4}{a^3})\; 0 + \frac{8y^6}{a^4} - \frac{4y^8}{a^6}$$

$$\frac{8y^6}{a^4} + \frac{16y^8}{a^6} - \frac{16y^{10}}{a^8} + \frac{16y^{12}}{a^{10}}$$

$$0 - \frac{20y^8}{a^6} + \frac{16y^{10}}{a^8} - \frac{16y^{12}}{a^{10}}$$

EXAMPLE II.

98. To reduce $\overline{1 - x - x^2}|^{\frac{1}{2}}$ into an *Infinite Series.*

Operation.

Operation.

$$1 - x - x^2 \left(1 - \frac{x}{2} - \frac{5x^2}{8} - \frac{5x^3}{16} - \text{\&c.} \right.$$

$$\begin{array}{rl}
 & 1 \\
2) & 0 - x - x^2 \\
 & -x + \frac{x^2}{4} \\
\hline
2 - x) & 0 \quad - \frac{5x^2}{4} \\
 & -\frac{5x^2}{4} + \frac{5x^3}{8} + \frac{25x^4}{64} \\
\hline
2 - x - \frac{5x^2}{4}) & 0 - \frac{5x^3}{8} - \frac{25x^4}{64} \\
 & -\frac{5x^3}{8} + \frac{5x^4}{16} + \frac{25x^5}{64} + \frac{25x^6}{256} \\
\hline
 & 0 - \frac{45x^4}{64} - \frac{25x^5}{64} - \frac{25x^6}{256} \\
 & \text{\&c.}
\end{array}$$

So that $\overline{1 - x - x^2}|^{\frac{1}{2}}$ is $= 1 - \frac{x}{2} - \frac{5x^2}{8} - \frac{5x^3}{16} - $ &c.

And, after the same manner may any such common Surd quantity be reduced into an *Infinite Series:* But, with much greater ease and expedition.

99. All sorts of *fractional* and *surd* Quantities may be reduced into *Infinite Series* by the celebrated

BINOMIAL THEOREM*,

which is this, *viz.* $\overline{P + PQ}|^{\frac{m}{n}} = P^{\frac{m}{n}} + \frac{m}{n} AQ + \frac{m-n}{2n} BQ + \frac{m-2n}{3n} CQ + \frac{m-3n}{4n} DQ + \frac{m-4n}{5n} EQ +$ &c. Wherein, P + PQ represents the Quantity whose Power is to be thrown into an *Infinite Series*; P the First term of that quantity, which, in general, must be the Greatest; Q the Other, or Rest of the terms, divided by the First; $\frac{m}{n}$ the Index of the power, whether it be affirmative or negative: and A = the First term of the *Series*, B = the Second, C = the Third, D = the Fourth, E = the Fifth, &c. that is, $A = P^{\frac{m}{n}}$, $B = \frac{m}{n} AQ$, $C = \frac{m-n}{2n} BQ$, $D = \frac{m-2n}{3n} CQ$, $E = \frac{m-3n}{4n} DQ$, &c.

The following Examples will explain, and shew the great Use of this curious and noble Theorem.

EXAMPLE I.

100. To reduce $\overline{a^2 + 4y^2}|^{\frac{1}{2}}$ into an *Infinite Series*.

Here, $P = a^2$, $Q = \frac{4y^2}{a^2}$, $m = 1$, $n = 2$, $A = a$,

* The Truth of this Theorem has been demonstrated by various Writers; the Proof of it is therefore here omitted.

$B = \frac{2y^2}{a}$, $C = -\frac{2y^4}{a^3}$, $D = \frac{4y^6}{a^5}$, $E = -\frac{10y^8}{a^7}$, &c. Therefore, $\overline{a^2 + 4y^2}|^{\frac{1}{2}} = a + \frac{2y^2}{a} - \frac{2y^4}{a^3} + \frac{4y^6}{a^5} - \frac{10y^8}{a^7}$, &c.

EXAMPLE II.

101. To reduce $\frac{1}{\overline{a^2 - y^2}|^{\frac{1}{2}}}$, that is, $\overline{a^2 - y^2}|^{-\frac{1}{2}}$, into an *Infinite Series.*

Here, $P = a^2$, $Q = -\frac{y^2}{a^2}$, $m = -1$, $n = 2$, $A = a^{-1} = \frac{1}{a}$, $B = \frac{y^2}{2a^3}$, $C = \frac{3y^4}{8a^5}$, $D = \frac{5y^6}{16a^7}$, $E = \frac{35y^8}{128a^9}$, &c. Therefore, $\overline{a^2 - y^2}|^{-\frac{1}{2}} = \frac{1}{a} + \frac{y^2}{2a^3} + \frac{3y^4}{8a^5} + \frac{5y^6}{16a^7} + \frac{35y^8}{128a^9}$, &c.

EXAMPLE III.

102. To reduce $\frac{1}{1 - x}$, that is, $\overline{1 - x}|^{-1}$, into an *Infinite Series.*

Here, $P = 1$, $Q = -x$, $m = -1$, $n = 1$,

$A = 1$, $B = x$, $C = x^2$, $D = x^3$, $E = x^4$, &c. Therefore, $\overline{1 - x}|^{-1} = 1 + x + x^2 + x^3 + x^4$, &c.

EXAMPLE IV.

103. To reduce $\frac{1}{1 + x}$, that is, $\overline{1 + x}|^{-1}$, into an *Infinite Series.*

Here, $P = 1$, $Q = x$, $m = -1$, $n = 1$, $A = 1$, $B = -x$, $C = x^2$, $D = -x^3$, $E = x^4$, &c. Therefore, $\overline{1 + x}|^{-1} = 1 - x + x^2 - x^3 + x^4$, &c.

104. BUT, we may often find the *Series* anſwering to a propoſed Quantity by the following

THEOREM,

viz. $\overline{P + PQ}|^{\frac{m}{n}} = P^{\frac{m}{n}} \times : 1 + \frac{m}{n} Q + \frac{m}{n} \times \frac{m - n}{2n} \times Q^2 + \frac{m}{n} \times \frac{m - n}{2n} \times \frac{m - 2n}{3n} Q^3 + \frac{m}{n} \times \frac{m - n}{2n} \times \frac{m - 2n}{3n} \times \frac{m - 3n}{4n} Q^4 +$ &c. (which, indeed, is the ſame as the former, though differently expreſſed,) with ſtill greater eaſe and expedition; for, in this, no previous deduction is required; and both the numerators and denominators of the fractions $\frac{m}{n}$, $\frac{m - n}{2n}$, $\frac{m - 2n}{3n}$, $\frac{m - 3n}{4n}$, &c. are Series of numbers in arithmetical progreſſion, which

have the ſame common difference n. This will appear by the following Examples.——But, *note*, the Former Theorem is, in general, beſt adapted to ſhew the *Law* of the Series.

Example I.

105. To reduce $\frac{1}{\overline{a^2 + x^2}|^{\frac{1}{2}}}$, that is, $\overline{a^2 + x^2}|^{-\frac{1}{2}}$, into an *Infinite Series*.

Here, $P = a^2$, $Q = \frac{x^2}{a^2}$, $m = -1$, $n = 2$.

Therefore, $\overline{a^2 + x^2}|^{-\frac{1}{2}} = \frac{1}{a} \times : 1 - \frac{x^2}{2a^2} + \frac{3.x^4}{2.4.a^4} - \frac{3.5.x^6}{2.4.6.a^6} + \frac{3.5.7.x^8}{2.4.6.8.a^8} - \text{\&c.} = \frac{1}{a} - \frac{x^2}{2a^3} + \frac{x^4}{8a^5} - \frac{5x^6}{16a^7} + \frac{35x^8}{128a^9} - \text{\&c.}$

Example II.

106. To reduce $\overline{a + x}|^{\frac{5}{3}}$ into an *Infinite Series*.

Here, $P = a$, $Q = \frac{x}{a}$, $m = 5$, $n = 3$. Therefore, $\overline{a + x}|^{\frac{5}{3}} = a^{\frac{5}{3}} \times : 1 + \frac{5.x}{3.a} + \frac{5.2.x^2}{3.6.a^2} +$

$$\frac{5.2.-1.x^3}{3.6.9.a^3} + \frac{5.2.-1.-4.x^4}{3.6.9.12.a^4} + \&c. = a^{\frac{5}{3}} + \frac{5a^{\frac{2}{3}}x}{3} + \frac{5x^2}{9a^{\frac{1}{3}}} - \frac{5x^3}{81a^{\frac{4}{3}}} + \frac{5x^4}{243a^{\frac{7}{3}}}, \&c.$$

Example III.

107. To reduce $\frac{b}{a+x}$, that is, $b \times \overline{a+x}|^{-1}$, into an *Infinite Series.*

Here, $P = a, Q = \frac{x}{a}, m = -1, n = 1$; and therefore, $\frac{m}{n}, \frac{m}{n} \times \frac{m-n}{2n}, \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n}, \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} \times \frac{m-3n}{4n}$, &c. will be -1 and $+1$ alternately; and confequently $b \times \overline{a+x}|^{-1}$ is $= b \times \frac{1}{a} \times : 1 - \frac{x}{a} + \frac{x^2}{a^2} - \frac{x^3}{a^3} + \frac{x^4}{a^4} - \&c. = \frac{b}{a} - \frac{bx}{a^2} + \frac{bx^2}{a^3} - \frac{bx^3}{a^4} + \frac{bx^4}{a^5} - \&c.$

Example IV.

108. To reduce $\overline{1-x}|^{\frac{1}{4}}$ into an *Infinite Series.*

Here, $P = 1, Q = -x, m = 1$ $n = 4$. Therefore, $\overline{1-x}|^{\frac{1}{4}} = 1 - \frac{1}{4}x + \frac{1.-3}{4.8}x^2 - \frac{1.-3.-7}{4.8.12}x^3$

$$+ \frac{1. - 3. - 7. - 11}{4. 8. 12. 16} x^4 - \&c. = 1 - \tfrac{1}{4} x - \frac{3}{4. 8} x^2 - \frac{3. 7}{4. 8. 12} x^3 - \frac{3. 7. 11}{4. 8. 12. 16.} x^4 - \&c.$$

Scholium.

The Sum of any Infinite geometrical ſeries decreaſing, is equal to the ſquare of the firſt term divided by the difference between the firſt and ſecond.

Thus, $a + x + \frac{x^2}{a} + \frac{x^3}{a^2} + \&c.$ is $= \frac{a^2}{a-x}$; and $a - x + \frac{x^2}{a} - \frac{x^3}{a^2} + \&c.$ is $= \frac{a^2}{a+x}$. For, (*art.* 94.) if a^2 be divided by $a - x$, and by $a + x$, the Quotients will be theſe infinite ſeries of Terms decreaſing in geometrical proportion continued.

CHAP. II.

Of finding the Fluent of a given Fluxion.

109. The Buſineſs of the *Direct* Method of Fluxions being to find the Fluxion of a given Fluent, or the Velocity with which a Variable

quantity flows at any point or term aſſigned; ſo the Buſineſs of the *Inverſe* Method of Fluxions is to determine the Variable quantity, or Fluent, from that Velocity or Fluxion being given. And this, in General, may be done by the following *Rules*, theſe being the converſe of thoſe delivered in *Part* 1. *Chap.* 2.*

Rule I.

110. To find the Fluent of a Simple Fluxion, or of that wherein there is no variable quantity and but One Fluxional letter.

Substitute the variable or flowing letter for its Fluxion: and you will have the Fluent required.

Thus, the Fluent of $a\dot{x}$ is $= ax$. (*art.* 14.)

Rule II.

111. To find the Fluent of a compound fluxional expreſſion conſiſting of the products of two or more flowing quantities drawn into their Fluxions; that is, which conſiſts of the Fluxion of each quantity drawn into the other or product of the reſt of the quantities.

* To treat at large on the different ways for finding the *Fluents* of the unbounded variety of Fluxional Expressions, would, by far, exceed the limits of an *introductory* Tract: this affair therefore cannot, with propriety, be handled here in so very extensive and copious a manner.

Multiply the flowing quantities together: and the Product is the Fluent required.

Thus, the fluent of $\dot{x}y + x\dot{y}$ is $= xy$; the Fluent of $\dot{x}yz + x\dot{y}z + xy\dot{z}$ is $= xyz$; and the Fluent of $\dot{v}xyz + v\dot{x}yz + vx\dot{y}z + vxy\dot{z}$ is $= vxyz$. (*art.* 15.)

Rule III*.

112. To find the Fluent of a Fraction like

$$\frac{\dot{x}y - x\dot{y}}{y^2}$$

Divide the laſt term in the numerator by the Fluxion of the Negative ſquare root of the denominator; then divide this quotient by the Affirmative ſquare root of the denominator: and you will have the Fluent required.

Thus, the Fluent of $\frac{\dot{x}y - x\dot{y}}{y^2}$ is $= \frac{x}{y}$. (*art.* 17.)

Rule IV.

113. To find the Fluent of an expreſſion compounded of different fluxionary terms connected together by the Signs + and —.

Find the ſeparate Fluents of the different terms; which connect together by the Signs of their reſpective Fluxions: and you will have the Fluent required.

* This *Rule* must be used with caution, as it is not applicable to fractional expressions in *general*.

Thus, the Fluent of $a\dot{x} + \dot{x}y + x\dot{y} - \frac{\dot{x}y - x\dot{y}}{y^2}$

is $= ax + xy - \frac{x}{y}$. (*art.* 19.)

RULE V.

114. To find the Fluent of an expreſſion which conſiſts of the Fluxion of a variable quantity drawn into any Power of that quantity contained any number of times *.

1°. In Simple Expreſſions,——Strike out the *fluxional* letter; add 1 to the Index of the Power; and divide by the Index thus increaſed: and you will have the Fluent required.

Thus, the Fluent of $2x^2\dot{x}$ is equal $\frac{2}{3}x^3$; the Fluent of $-\frac{\dot{x}}{x^2}$, that is, of $-x^{-2}\dot{x}$, is $= -x^{-2+1}$ divided by $-2+1$, that is, $= \frac{-x^{-1}}{-1} = x^{-1} = \frac{1}{x}$. And, Univerſally, the Fluent of $mx^{m-1}\dot{x}$ is $= x^m$; and the Fluent of $\frac{m}{n}x^{\frac{m}{n}-1}\dot{x}$ is $= x^{\frac{m}{n}}$.

2°. In Compound Expreſſions,—where the *fluxionary* part is equal, or in an invariable ratio, to the Fluxion of the quantity under the vinculum, Add 1 to the Index of the Power; and divide by

* The *Rule* fails when the Index of the Power is -1. To find the Fluent then, see *Rule* 6.

the Fluxion of the quantity under the vinculum, drawn into the Index of the Power thus increaſed: and you will have the Fluent required.

Thus, the Fluent of $\overline{x + x^2}|^2 \times \overline{3\dot{x} + 6x\dot{x}}$ is $= \dfrac{\overline{x + x^2}|^3 \times \overline{3\dot{x} + 6x\dot{x}}}{3 \times \dot{x} + 2x\dot{x}} = \overline{x + x^2}|^3$; and the Fluent of $-\frac{2}{3} \times \overline{x - a}|^{-\frac{5}{3}} \times \dot{x}$ is $= \dfrac{-\frac{2}{3} \times \overline{x-a}|^{-\frac{5}{3}+1} \times \dot{x}}{-\frac{5}{3} + 1 \times \dot{x}} = \dfrac{-\frac{2}{3} \times \overline{x-a}|^{-\frac{2}{3}}\dot{x}}{-\frac{2}{3}\dot{x}} = \overline{x - a}|^{-\frac{2}{3}}$. (*art.* 20)

Rule VI.

115. To find the Fluent of a compound fluxional expreſſion, like $\dfrac{b}{a + x}\dot{x}$; or $\dfrac{b\dot{x}}{\overline{a^2 + x^2}|^{\frac{1}{2}}}$; *&c.*

1°. Throw the expreſſion into an infinite Series; and find the fluent of the Series by the foregoing Rules: and you will have the Fluent required.

Thus, to find the Fluent of $\dfrac{b}{a + x}\dot{x}$; throw the expreſſion into a Series; which (*art.* 94.) is $= \dfrac{b\dot{x}}{a} - \dfrac{bx\dot{x}}{a^2} + \dfrac{bx^2\dot{x}}{a^3} - \dfrac{bx^3\dot{x}}{a^4} +$ *&c.* and then find the Fluent of this Series; which, by the preceding

Rules, is $= \frac{bx}{a} - \frac{bx^2}{2a^2} + \frac{bx^3}{3a^3} - \frac{bx^4}{4a^4} +$ &c. and is the Fluent required.

And, to find the Fluent of $\frac{b\dot{x}}{\overline{a^2+x^2}|^{\frac{1}{2}}}$; throw the expreſſion into a Series; which, (*art.* 105.) is $= b\dot{x} \times : \frac{1}{a} - \frac{x^2}{2a^3} + \frac{3x^4}{8a^5} - \frac{5x^6}{16a^7} +$ &c. $=$ $\frac{b\dot{x}}{a} - \frac{bx^2\dot{x}}{2a^3} + \frac{3bx^4\dot{x}}{8a^5} - \frac{5bx^6\dot{x}}{16a^7} +$ &c. Now, the Fluent of this Series, by the foregoing Rules, is $= \frac{bx}{a} - \frac{bx^3}{6a^3} + \frac{3bx^5}{40a^5} - \frac{5bx^7}{112a^7} +$ &c. which is the Fluent required.

2°. Or, Becauſe (*art.* 21.) the Fluxion of the Hyperbolic Logarithm of any quantity is equal to the Fluxion of that quantity, divided by the quantity itſelf; therefore,

The Fluent of $b \times \frac{\dot{x}}{a+x}$ is $= b \times$ Hyp. Log. of $a + x$. For, the Fluxion of $a + x$ is $\dot{x}$, which divided by $a + x$ is $\frac{\dot{x}}{a+x}$.

And the Fluent of $b \times \frac{\dot{x}}{\overline{a^2+x^2}|^{\frac{1}{2}}}$ is $= b \times$ Hyp. Log. of $x + \overline{a^2+x^2}|^{\frac{1}{2}}$. For, the Fluxion of $x + \overline{a^2+x^2}|^{\frac{1}{2}}$ is $\dot{x} + \frac{x\dot{x}}{\overline{a^2+x^2}|^{\frac{1}{2}}} =$

$$\frac{\dot{x} \times \overline{a^2 + x^2}|^{\frac{1}{2}} + x\dot{x}}{\overline{a^2 + x^2}|^{\frac{1}{2}}} = \frac{\dot{x}}{\overline{a^2 + x^2}|^{\frac{1}{2}}} \times : x + \overline{a^2 + x^2}|^{\frac{1}{2}},$$

which divided by $x + \overline{a^2 + x^2}|^{\frac{1}{2}}$ is $\frac{\dot{x}}{\overline{a^2 + x^2}|^{\frac{1}{2}}}$. &c.

Scholium.

116. Though the Fluxion of any Fluent, how much ſoever compounded it be, may be accurately found; yet, the Fluents of compounded Fluxional expreſſions cannot always be had in finite terms.

117. Though no Fluent can have more than One Fluxion; yet, a Fluxion may have an Infinite number of Fluents. Thus, for Example, the Fluent of $\dot{x}$ may be either x or $x \pm a$; wherein, a repreſents any invariable quantity whatſoever.—Now, to find a, when it muſt be added to or taken from the Fluent x, is called *correcting* the Fluent: And to effect this; in any Equation, after having obtained the Fluent of each ſide by the foregoing Rules; make the variable letter in either of them vaniſh, or equal to nothing; and ſubſtitute for the variable letter in the other, ſuch a determinate or invariable value as it is then known to have: or, for the variable quantities, write ſuch invariable or fixed values as they are reſpectively known to have at any particular point or term. Then, if we ſubtract the ſides of this new Equation from the correſponding Fluents before found; the remaining Fluents will be Contemporary, or always equal to each other; and conſequently, we ſhall then have the correct Fluent ſought deter-

mined.—This affair may, perhaps, be better underſtood by giving the following Examples.

Example 1.

To find the correct Fluent of $\dot{y} = 2x\dot{x}$.

The Fluent of this equation, by *art.* 114. is $y = x^2$. Now, when $y = 0$, if $x = 0$; then, $y - 0 = x^2 - 0$, or $x = y^2$: therefore, the Fluent firſt found needs no correction.

Example 2.

To find the correct Fluent of $a\dot{x} - 2x\dot{x} = 2y\dot{y}$.

The Fluent of this equation, by *art.* 114. is $ax - x^2 = y^2$. Now, when y ends, or when $y = 0$, if x be $= a$; then, ſubſtituting 0 for y, and a for x, the fluential equation will become $a^2 - a^2 = 0$, that is $0 = 0$: therefore, the Fluent firſt found needs no correction.

Example 3.

To find the Correct Fluent of $\dot{z} = a\dot{y}$.

The Fluent of this equation, by *art.* 110. is $z = ay$. Now, if y be $= b$ when z is $= 0$; then, by writing in this fluential equation 0 for z and b for y, it will become $0 = ab$: therefore, ay is always ab greater than z; and conſequently, the Fluent corrected is $z = ay - ab$.

Example 4.

To find the Correct Fluent of $\dot{y} = -\overline{a+x}|^{-3} \times 2\dot{x}$.

The Fluent of this equation, by *art.* 114. is $y = \frac{-\overline{a+x}|^{-2} \times 2\dot{x}}{-2\dot{x}} = \overline{a+x}|^{-2} = \frac{1}{\overline{a+x}|^2}$. Now, if $y = 0$ when $x = 0$; then, this fluential equation will become $0 = \frac{1}{a^2}$: and therefore, y is always less than $\frac{1}{\overline{a+x}|^2}$ by the quantity $\frac{1}{a^2}$: consequently, the Contemporary Fluents will be $y = \frac{1}{\overline{a+x}|^2} - \frac{1}{a^2}$.

Example 5.

To find the Correct Fluent of $\dot{z} = b \times \frac{\dot{x}}{\overline{a^2+x^2}|^{\frac{1}{2}}}$

The Fluxion of the Hyp. Log. of $x + \overline{a^2+x^2}|^{\frac{1}{2}}$ is $= \frac{\dot{x}}{\overline{a^2+x^2}|^{\frac{1}{2}}}$ (*art.* 21. or 115.) therefore, the Fluent, of $\dot{z} = b \times \frac{\dot{x}}{\overline{a^2+x^2}|^{\frac{1}{2}}}$ is $z = b \times$ Hyp. Log. of $x + \overline{a^2+x^2}|^{\frac{1}{2}}$. Now, when $z = 0$, if x be likewise $= 0$, this Fluent will then become

$o = b \times$ Hyp. Log. of a; which ſubtracted from the ſaid Fluent, makes the Correct Fluent, or *true* value of $z = b \times$ Hyp. Log. of $x + \overline{a^2 + x^2})^{\frac{1}{2}}$ — $b \times$ Hyp. Log. of a (which, by the Nature of Logarithms, is,) $= b \times$ Hyp. Log. of $\frac{x+\overline{a^2+x^2})^{\frac{1}{2}}}{a}$.

Example 6.

To find the Correct Fluent of $a\dot{x} - bx\dot{x} = y\dot{y}$:

The Fluent of this equation (*art.* 114.) is $ax - \frac{1}{2}bx^2 = \frac{1}{2}y^2$. Now, if $x = c$ when $y = d$; then, ſubſtituting c for x and d for y, the equation will be $ac - \frac{1}{2}bc^2 = \frac{1}{2}d^2$; and therefore, by ſubtracting the correſponding ſides of this equation from the above, we ſhall have the Correct or Contemporary Fluents $ax - \frac{1}{2}bx^2 - ac + \frac{1}{2}bc^2 = \frac{1}{2}y^2 - \frac{1}{2}d^2$.

CHAP. III.

Of finding the Length of a Curve Line.

Fig. 57. 58. 118. In Curves referred to an Axis, (*fig.* 57.) let cb be ſuppoſed indefinitely near and parallel to the ordinate CB, and Bn equal and parallel to Cc the Increment of the abſciſs AC: And in curves

referred to a fixed or central Point, (*fig.* 58.) let bC be ſuppoſed indefinitely near to BC, and the indefinitely little circular arch Bn be deſcribed with the ordinate or radius CB. Put AC (*fig.* 57.) $= x$, CB $= y$, curve AB $= z$; B$n = x'$, $nb = y'$, and B$b = z'$: then, (the Increment Bb being conſidered as a little right line,) by 47 E. 1. B$b = \overline{Bn^2 + nb^2}\vert$, that is, $z' = \overline{x'^2 + y'^2}\vert^{\frac{1}{2}}$, or (*art.* 7.) $\dot{z} = \overline{\dot{x}^2 + \dot{y}^2}\vert^{\frac{1}{2}}$. Put the ordinate CB (*fig.* 58.) $=$ y, curve CPB $= z$, B$n = x'$, $nb = y'$, and B$b =$ z': then, (becauſe Bb may be conſidered as an indefinitely ſmall right line, and Bn as a little right line perpendicular to Cb,) as before, $z' = \overline{x'^2 + y'^2}\vert^{\frac{1}{2}}$ or $\dot{z} = \overline{\dot{x}^2 + \dot{y}^2}\vert^{\frac{1}{2}}$. And this is a General Expreſſion for the Fluxion of the Length of any Curve Line whatſoever.* Now, by help of the Equation or Properties of the given Curve whoſe Length is required, we may find the value of $\dot{x}^2$ in terms of $\dot{y}^2$, or of $\dot{y}^2$ in terms of $\dot{x}^2$; and then, by ſubſtitution, $\dot{x}^2$ or $\dot{y}^2$ in this General Expreſſion will be exterminated: and by finding the Fluent of the reſulting equation, we ſhall have the value of z, or the Length of the Curve required.

Example I.

119. To find the Length of the Curve AB $= z$, whoſe Equation (putting the given line AG $=$ $\frac{4}{3}a$, GC $= x$, and CB $= y$,) is $2 \times \overline{a^2 + x^2}\vert^{\frac{3}{2}} = 3a^2 y$.

* The same expression may be derived, without the help of *Increments*, from *art.* 24 and 38.

The Fluxion of this equation is $3 \times \overline{a^2 + x^2}|^{\frac{1}{2}} \times 2x\dot{x} = 3a^2\dot{y}$; therefore, $\dot{y} = \frac{2x\dot{x}}{3a^2} \times 3 \times \overline{a^2 + x^2}|^{\frac{1}{2}} = \frac{2x\dot{x}}{a^2} \times \overline{a^2 + x^2}|^{\frac{1}{2}}$, and $\dot{y}^2 = \frac{4x^2\dot{x}^2}{a^4} \overline{a^2 + x^2} = \frac{4a^2x^2\dot{x}^2 + 4x^4\dot{x}^2}{a^4}$; which substituted for $\dot{y}^2$, makes $\dot{z} = \overline{\dot{x}^2 + \dot{y}^2}|^{\frac{1}{2}}$ (*art.* 118.) $= \frac{\overline{a^4\dot{x}^2 + 4a^2x^2\dot{x}^2 + 4x^4\dot{x}^2}|^{\frac{1}{2}}}{a^4} = \frac{a^2\dot{x} + 2x^2\dot{x}}{a^2}$ = the Fluxion of the Curve AB, whose Fluent is $z = \frac{a^2x + \frac{2}{3}x^3}{a^2} = x + \frac{2x^3}{3a^2}$ = the Length of the Curve AB required.

EXAMPLE II.

120. To find the Length of the common *Cycloid.**

Fig. 60. Put OA or OF the radius of the generating circle $= a$, abscifs AC $= x$, ordinate CB $= y$, CG $= s$, and arch AB $= z$; then, (as was found in *art.* 35.) $\dot{y} = \frac{2a - x}{s}\dot{x}$, and therefore $\dot{y}^2 = \frac{\overline{2a - x}|^2}{s^2}\dot{x}^2$; which substituted for $\dot{y}^2$, makes the general

* See the generation of this Curve, *art.* 35. *note.*

expreſſion for the Fluxion of the Curve (*art.* 118.) $\dot{z} = \overline{\dot{x}^2 + \dot{y}^2}\big|^{\frac{1}{2}} = \overline{\dot{x}^2 + \frac{\overline{2a-x}\big|^2}{s^2}\,\dot{x}^2}\Big|^{\frac{1}{2}} = \overline{\frac{s^2\dot{x}^2 + 4a^2\dot{x}^2 - 4ax\dot{x}^2 + x^2\dot{x}^2}{s^2}}\Big|^{\frac{1}{2}}$, that is, (becauſe by 35 E. 3. $GC^2 = AC \times CF$, or $s^2 = 2ax - x^2$,) $\dot{z} = \overline{\frac{4a^2\dot{x}^2 - 2ax\dot{x}^2}{2ax - x^2}}\Big|^{\frac{1}{2}} = \overline{\frac{2a\dot{x}^2}{x}}\Big|^{\frac{1}{2}} = \overline{2a}\big|^{\frac{1}{2}} \times x^{-\frac{1}{2}}\dot{x}$; and the Fluent of this is $z = \overline{2a}\big|^{\frac{1}{2}} \times 2x^{\frac{1}{2}} = 2 \times \overline{2ax}\big|^{\frac{1}{2}} =$ twice the chord AG; (for, the triangles FAG and GAC being ſimilar, FA : AG :: GA : AC, or $\overline{FA \times AC}\big|^{\frac{1}{2}} = AG$, that is, $\overline{2ax}\big|^{\frac{1}{2}} = AG$.) Whence, by writing $2a$ for x, the Length of the Semicycloid AD appears to be equal to twice the diameter AF of it's generating Semicircle.

Example III.

121. To find the Length of a *Parabola.*

Put the parameter $= a$, abſciſ AC $= x$, ordinate CB $= y$, and curve AB $= z$; then (*art.* 28.) $\dot{x} = \frac{2y\dot{y}}{a}$, and therefore $\dot{x}^2 = \frac{4y^2\dot{y}^2}{a^2}$; which ſubſtituted for $\dot{x}^2$, makes $\dot{z} = \overline{\dot{x}^2 + \dot{y}^2}\big|^{\frac{1}{2}}$ (the general expreſſion for the Fluxion o the Length of the *Fig.* 61.

Curve, *art.* 118.) $= \overline{\frac{4y^2\dot{y}^2}{a^2} + \dot{y}^2}\Big|^{\frac{1}{2}} = \frac{\dot{y}}{a} \times \overline{a^2 + 4y^2}\big|^{\frac{1}{2}}$; which, thrown into an Infinite Series, (*art.* 100.) is $= \frac{\dot{y}}{a} \times : a + \frac{2y^2}{a} - \frac{2y^4}{a^3} + \frac{4y^6}{a^5} -$ &c. that is, $\dot{z} = \dot{y} + \frac{2y^2\dot{y}}{a^2} - \frac{2y^4\dot{y}}{a^4} + \frac{4y^6\dot{y}}{a^6} -$ &c. And the Fluent of this Series is $z = y + \frac{2y^3}{3a^2} - \frac{2y^5}{5a^4} + \frac{4y^7}{7a^6} -$ &c. $=$ the Length of the Curve AB required.

Or, The above $\dot{z} = \frac{\dot{y}}{a} \times \overline{a^2 + 4y^2}\big|^{\frac{1}{2}}$ is $= \frac{\dot{y} \times \overline{a^2 + 4y^2}}{a \times \overline{a^2 + 4y^2}\big|^{\frac{1}{2}}} = \frac{a^2y\dot{y} + 4y^3\dot{y}}{a \times \overline{a^2y^2 + 4y^4}\big|^{\frac{1}{2}}} = \frac{\frac{1}{2}a^2y\dot{y} + 4y^3\dot{y}}{a \times \overline{a^2y^2 + 4y^4}\big|^{\frac{1}{2}}} + \frac{\frac{1}{2}a^2\dot{y}}{a \times \overline{a^2 + 4y^2}\big|^{\frac{1}{2}}} = \frac{1}{a} \times \overline{a^2y^2 + 4y^4}\big|^{-\frac{1}{2}} \times \overline{\frac{1}{2}a^2y\dot{y} + 4y^3\dot{y}} + \frac{1}{4}a \times \frac{\dot{y}}{\overline{\frac{1}{4}a^2 + y^2}\big|^{\frac{1}{2}}}$. Now, (*art.* 114.) the Fluent of the first of these two terms is $= \frac{1}{2a} \times \overline{a^2y^2 + 4y^4}\big|^{\frac{1}{2}} = \frac{y}{a} \times \overline{\frac{1}{4}a^2 + y^2}\big|^{\frac{1}{2}}$; and the Fluent of the last of the said two terms (*art.* 115.) is $= \frac{1}{4}a \times$ Hyp. Log. of $y + \overline{\frac{1}{4}a^2 + y^2}\big|^{\frac{1}{2}}$; therefore, $z = \frac{y}{a} \times \overline{\frac{1}{4}a^2 + y^2}\big|^{\frac{1}{2}} + \frac{1}{4}a \times$ Hyp.

Log. of $y + \overline{\frac{1}{4}a^2 + y^2}|^{\frac{1}{2}}$. But, since when z and y vanish or become $= 0$, (as at the vertex A,) this Fluent becomes $= \frac{1}{4} a \times$ Hyp. Log. of $\frac{1}{2} a$; therefore the said Fluent being Corrected (*art.* 117. *Ex.* 5.) makes the *true* value of z or the Length of the Curve $AB = \frac{y}{a} \times \overline{\frac{1}{4}a^2 + y^2}|^{\frac{1}{2}} + \frac{1}{4} a \times$ Hyp. Log. of $\overline{y + \overline{\frac{1}{4}a^2 + y^2}|^{\frac{1}{2}}} - \frac{1}{4} a \times$ Hyp. Log. of $\frac{1}{2} a = \frac{y}{a} \times \overline{\frac{1}{4}a^2 + y^2}|^{\frac{1}{2}} + \frac{1}{4} a \times$ Hyp. Log. of $\dfrac{y + \overline{\frac{1}{4}a^2 + y^2}|^{\frac{1}{2}}}{\frac{1}{2}a}$, by the Nature of Logarithms.

Example IV.

122. To find the Length of any Arch of a *Circle.*

Put the radius $EA = a$, abscifs $AC = x$, ordinate $CB = y$, arch $AB = z$: then, (*art.* 27.) $\dot{x} = \dfrac{y\dot{y}}{a - x} =$ (because $a - x = CE = \overline{EB^2 - BC^2}|^{\frac{1}{2}} = \overline{a^2 - y^2}|^{\frac{1}{2}}$,) $\dfrac{y\dot{y}}{\overline{a^2 - y^2}|^{\frac{1}{2}}}$; therefore $\dot{x}^2 = \dfrac{y^2\dot{y}^2}{a^2 - y^2}$; Fig. 62.

which substituted for $\dot{x}^2$, makes $\dot{z} = \overline{\dot{x}^2 + \dot{y}^2}|^{\frac{1}{2}}$ (the general expression for the Fluxion of the Length of the Curve, *art.* 118.) $= \overline{\dfrac{y^2\dot{y}^2}{a^2 - y^2} + \dot{y}^2}|^{\frac{1}{2}} =$

$\left(\dot{y}^2 - \frac{y^2 t^2}{r^2 + t^2}\right)$

$\frac{a\dot{y}}{\overline{a^2-y^2}|^{\frac{1}{2}}} = a\dot{y} \times \overline{a^2-y^2}|^{-\frac{1}{2}}$; which thrown into an Infinite Series (*art.* 101.) is $\dot{z} = a\dot{y} \times : \frac{1}{a} + \frac{y^2}{2a^3} + \frac{3y^4}{8a^5} + \frac{5y^6}{16a^7} + \frac{35y^8}{128a^9} + \frac{63y^{10}}{256a^{11}} + \frac{231y^{12}}{1024a^{13}} + \frac{429y^{14}}{2048a^{15}} +$ &c. $= \dot{y} + \frac{y^2\dot{y}}{2a^2} + \frac{3y^4\dot{y}}{8a^4} + \frac{5y^6\dot{y}}{16a^6} + \frac{35y^8\dot{y}}{128a^8} + \frac{63y^{10}\dot{y}}{256a^{10}} + \frac{231y^{12}\dot{y}}{1024a^{12}} + \frac{429y^{14}\dot{y}}{2048a^{14}} +$ &c. And the Fluent of this Series (*art.* 114.) is $z = y + \frac{y^3}{6a^2} + \frac{3y^5}{40a^4} + \frac{5y^7}{112a^6} + \frac{35y^9}{1152a^8} + \frac{63y^{11}}{2816a^{10}} + \frac{231y^{13}}{13312a^{12}} + \frac{143y^{15}}{10240a^{14}}$ + &c. = the Length of the Arch AB.

Now, if we ſuppoſe the radius EA $= a = 1$, and the ∠ AEB = 30°. then, (becauſe the ſine of any arch is equal to half the chord of twice that arch, and the chord of 60°. is equal to the radius,) the ſine, or ordinate CB $= y$, will be $= \frac{1}{2}$; and therefore, the terms of the above Fluent being reduced into Decimal Fractions, and placed under one another, will ſtand thus: viz.

.500000000 &c.
.020833333
.002343750
.000348772
.000059339
.000010923
.000002118
.000000426 &c.

The ſum of which is	$\overline{\cdot 5235987}$ &c. =	the Length of the Arch AB:
which × by	 12	
is	$\overline{6.283185}$ &c. =	the Periphery of a Circle whoſe Radius is 1;
therefore	3,141592 &c. =	the Periphery of a Circle whoſe Diameter is 1.

EXAMPLE V.

123. To find the Length of any Arch of *Archimedes's Spiral**.

Put CA the radius of the generating circle $= b$, and ARA the circumference of it $= a$; alſo, put the ordinate CB $= y$, the length of the required arch CPB $= z$, and a circular arch whoſe radius is the ordinate CB $= x$. Then, (as was found in *art.* 39.) $\dot{x} = \frac{ay\dot{y}}{b^2}$, and therefore $\dot{x}^2 = \frac{a^2y^2\dot{y}^2}{b^4}$; which ſubſtituted for $\dot{x}^2$, makes the general expreſſion for the Fluxion of the Curve, viz. $\dot{z} = \overline{\dot{x}^2 + \dot{y}^2}|^{\frac{1}{2}}$ (*art.* 118.) $= \overline{\frac{a^2y^2\dot{y}^2}{b^4} + \dot{y}^2}|^{\frac{1}{2}} = \frac{\dot{y}}{b^2} \times \overline{a^2y^2 + b^4}|$ *Fig.* 63.

$$= \frac{\dot{y} \times \overline{a^2y^2 + b^4}}{b^2 \times \overline{a^2y^2 + b^4}|^{\frac{1}{2}}} = \frac{a^2y\dot{y} + b^4y\dot{y}}{b^2 \times \overline{a^2y^4 + b^4y^2}|^{\frac{1}{2}}} =$$

* See how this Curve is generated, *art.* 39. *note.*

$$\frac{a^2 y^3 \dot{y} + \frac{1}{2} b^4 y\dot{y}}{b^2 \times \overline{a^2 y^4 + b^4 y^2}|^{\frac{1}{2}}} + \frac{\frac{1}{2} b^4 \dot{y}}{b^2 \times \overline{a^2 y^2 + b^4}|^{\frac{1}{2}}} = \frac{1}{b^2} \times$$

$$\overline{a^2 y^4 + b^4 y^2}|^{-\frac{1}{2}} \times \overline{a^2 y^3 \dot{y} + \frac{1}{2} b^4 y\dot{y}} + \frac{b^2}{2a} \times \frac{a\dot{y}}{\overline{a^2 y^2 + b^4}|^{\frac{1}{2}}}$$

Now, the Fluent of the firſt of theſe two terms is found by *art.* 114. $= \frac{1}{2b^2} \times \overline{a^2 y^4 + b^4 y^2}|^{\frac{1}{2}}$; and the Fluent of the laſt is found by *art.* 115. $= \frac{b^2}{2a} \times$ Hyp. Log. of $ay + \overline{a^2 y^2 + b^4}|^{\frac{1}{2}}$; therefore $z = \frac{1}{2b^2} \times \overline{a^2 y^4 + b^4 y^2}|^{\frac{1}{2}} + \frac{b^2}{2a} \times$ Hyp. Log. of $ay + \overline{a^2 y^2 + b^4}|^{\frac{1}{2}}$: but, when z and y are $= 0$, as at the point C, this equation becomes $0 = \frac{b^2}{2a} \times$ Hyp. Log. of b^2; and therefore, the Fluent Corrected, (*art.* 117.) makes the *true* value of $z = \frac{1}{2b^2} \times \overline{a^2 y^4 + b^4 y^2}|^{\frac{1}{2}} + \frac{b^2}{2a} \times$ Hyp. Log. of $ay + \overline{a^2 y^2 + b^4}|^{\frac{1}{2}}$ $- \frac{b^2}{2a} \times$ Hyp. Log. of b^2 = (by the Nature of Logarithms; the Difference of the Logarithms of any two Numbers being equal to the Logarithm of their Quotient;) $\frac{1}{2b^2} \times \overline{a^2 y^4 + b^4 y^2}|^{\frac{1}{2}} + \frac{b^2}{2a} \times$ Hyp. Log. of $\frac{ay + \overline{a^2 y^2 + b^4}|^{\frac{1}{2}}}{b^2}$ = the Length of the

Arch CPB required. And therefore, by ſubſtituting the radius b for the ordinate y, we ſhall have the Length of the whole Spiral CPBA $= \frac{1}{2} \times \overline{a^2 + b^2}|^{\frac{1}{2}}$ $+ \frac{b^2}{2a} \times$ Hyp. Log: of $\frac{a + \overline{a^2 + b^2}|^{\frac{1}{2}}}{b}$.

CHAPTER IV.

Of finding the Areas of Curvilineal Spaces.

124. In Curves whoſe ordinates are referred to an Axis (*fig.* 64) let bc be conceived indefinitely near and parallel to the perpendicular ordinate BC, and Bn equal and parallel to Cc the Increment of the abſciſs AC : then, becauſe bn bears no aſſignable ratio to BC, bc may be taken as equal to BC or nc; and the trapezium BCcb as equal to the parallelogram BCcn: but, BCcb is the Moment or Increment of the curvilineal ſpace ABC, that is, (if we put AC $= x$, CB $= y$, and C$c = x'$,) the Moment or Increment of the Space ABC is $= yx'$ and therefore, (*art.* 7.) the Fluxion of it is $= y\dot{x}$. —In like manner, in Spirals, or thoſe Curves whoſe ordinates are referred to a fixed or central Point (*fig.* 65.) let bC be conceived indefinitely near to the ordinate BC, and the little circular arch Bn (whoſe radius is CB,) be ſuppoſed a little right line *Fig.* 64, 65.

perpendicular to bC: then, bn having lefs than any affignable ratio to nC, BCn may be confidered as equal to BCb the Moment or Increment of the curvilineal Space BPCB; that is, (if we put $CB = y$, and $Bn = x'$,) the Moment or Increment of the Space CPBC is $= \frac{1}{2}\ yx'$, or it's Fluxion $= \frac{1}{2}\ y\dot{x}$.

Or, Let the curvilineal fpace AEI and parallelogram AG (*fig.* 64.) be generated by the perpendicular and indefinite right line AF moving with a parallel motion from A along the axis AE; then, it is evident, the curvilineal fpace will increafe flower or flow with a lefs degree of velocity, than the parallelogram, before the faid generating line arrives at the term CB; and afterwards fafter, or with a greater degree of velocity: therefore, at the faid term, they will flow with one and the fame degree of velocity: that is, at the term CB, the Fluxions of the curvilineal fpace and parallelogram will be equal: But, it is plain, the Fluxion of the parallelogram, at the term CB, is equal to DA or BC drawn into the Fluxion of AC; therefore, the Fluxion of the curvilineal fpace ACB is equal to the ordinate CB drawn into the Fluxion of the abfcifs AC; that is, (putting $AC = x$, and $CB = y$,) the Fluxion of the curvilineal Space ACB is $= y\dot{x}$.—In *fig.* 65. let the curvilineal fpace CPIC be generated by the variable right line CF turning round the center C; and, at the fame time, let the fector CDGC be defcribed by the radius CD; then, it is plain, before the line CF comes to be in the fituation CB, the fpiral fpace will increafe flower or flow with a lefs degree of velocity, than the circular fpace or fector CDG; and afterwards fafter, or with a greater degree of velocity: therefore at

the term CB, they will increase or flow with an equal degree of velocity. But, it is evident, the velocity with which the sector enlarges, is equal to half it's radius drawn into the velocity with which it's arch is described; therefore, the velocity with which the curvilineal space CPBC is increased, at the term CB, is equal to $\frac{1}{2}$ CB drawn into the velocity of the point D or B moving along the arch DG at the point B; that is, the Fluxion of the said curvilineal space is equal to $\frac{1}{2}$ CB drawn into the Fluxion of the circular arch DB; or (putting the ordinate CB $= y$, and arch DB $= x$,) the Fluxion of the curvilineal Space CPBC is $= \frac{1}{2} y\dot{x}$; as before.

125. Wherefore, when the Curve is referred to an Axis, (*fig.* 64.) find the value of y in terms of x, by help of the Equation of the given Curve, which multiply by $\dot{x}$; or, find the value of in terms of $\dot{y}$, which multiply by y: Then, the Fluent of the resulting fluxional expression will express the Area (or quadrature) of the curvilineal Space ABC required.——And, when the Curve is referred to a fixed or central Point C, (*fig.* 65.) find the value of $\dot{x}$ in terms of y, from the properties of the given curve: then multiply this value of $\dot{x}$ by $\frac{1}{2}y$, and find the Fluent; and it will give the Area of the Space required.

Example I.

126. To find the Area of the Space ABCA; the Curve AB being a *Parabola*. *Fig.* 66.

Put the abscis AC $= x$, ordinate CB $= y$, and

the parameter $= 1$. Then, by the nature of the curve, $x = y^2$, or $x^{\frac{1}{2}} = y$; therefore, $y\dot{x}$ (the Fluxion of the Area, *art.* 124.) $= x^{\frac{1}{2}}\dot{x}$; the Fluent of which is $\frac{2}{3}x^{\frac{3}{2}} =$ (by ſubſtituting y for $x^{\frac{1}{2}}$ it's value,) $\frac{2}{3}xy =$ the Area required.

Or, The Fluxion of the above equation of the curve, viz. of $x = y^2$, is $\dot{x} = 2y\dot{y}$; which, multiplied by y, makes the Fluxion of the Area, viz. $y\dot{x}$ (*art.* 124.) $= 2y^2\dot{y}$; the Fluent of which is $\frac{2}{3}y^3 =$ (by writing x for y^2 it's value,) $\frac{2}{3}xy$; as before.

Corollary.

The Area of any Parabolic Space ABCA is equal to two-third-parts of it's circumſcribing Parallelogram ADBC.

Example II.

Fig. 67. 127. To find the Area of the Space ABCG; the property of the Curve AB being ſuch, that it's Subtangent CT is invariable, or always of the ſame value. (See *art* 80.)

Put the given Subtangent CT $= a$, GA $= b$, GC $= x$, and CB $= y$. Then, (*art.* 25.) $a = \frac{\dot{x}y}{\dot{y}}$; therefore, $\dot{x} = \frac{a\dot{y}}{y}$; which multiplied by y makes the Fluxion of the Space, viz. $y\dot{x}$ (*art.* 124.) $= a\dot{y}$;

and the Fluent of this Fluxion is ay: But, when the area of the ſpace is $= 0$. or $y = b$, this expreſſion for the Fluent becomes $= ab$; and therefore, (*art.* 117. *Ex.* 3.) the Fluent corrected is $ay - ab = \overline{y - b} \times a =$ the Area of the Space ABCG required.

EXAMPLE III.

128. To find the Area of the Space CPBC; the Curve CPBA being the *Spiral* of *Archimedes*.* *Fig.* 63.

Put the circumference of the generating circle ARA $= a$, and it's radius CA $= b$; alſo, put the ordinate CB $= y$. Now, (as was found in *art.* 39.) $\dot{x} = \frac{ay\dot{y}}{b^2}$; which equation multiplied by $\frac{1}{2}y$ makes the Fluxion of the Space, viz. $\frac{1}{2}y\dot{x}$ (*art.* 124.) $= \frac{ay\dot{y}}{b^2} \times \frac{1}{2}y = \frac{ay^2\dot{y}}{2b}$; the Fluent of which is $\frac{ay^3}{6b^2}$ $=$ the Area of the Space required. And therefore by ſubſtituting b for y, we have the Area of the whole ſpiral Space CPBAC $= \frac{1}{6}ab$; which, becauſe the area of a circle is equal to it's periphery drawn into half the radius, is $= \frac{1}{3}$ of the Area of the generating Circle.

EXAMPLE. IV.

129. To find the Area of the Space CPBC; the Curve CPB being the *Logarithmic Spiral*.† *Fig.* 68.

* See the generation of this Curve, *art.* 39. *note.*
† See the generation of this Curve, *art.* 40. *note.*

Put the ordinate CB $= y$; the length of the curve CPB $= z$; a circular arch, whose radius is the ordinate CB, $= x$; and let two given quantities a and b, be to each other in the ratio of y to z.

Then, (as was found in *art.* 40.) $\dot{x} = \dot{y} \times \frac{\overline{b^2 - a^2}|^{\frac{1}{2}}}{a}$;

which, multiplied by $\frac{1}{2}y$, makes $\frac{1}{2}y\dot{x}$ (the general expression for the Fluxion of the Area, *art.* 124.) $= \frac{\overline{b^2 - a^2}|^{\frac{1}{2}}}{2a} y\dot{y}$; the Fluent of which is $\frac{\overline{b^2 - a^2}|^{\frac{1}{2}}}{4a} y^2 =$ the Area of the Space CPBC required.

Example V.

Fig. 69, 70. 130. To find the Area of the Space CHBRC; the Curve being a *Spiral* generated by a point moving uniformly along the semicircle CDA, from C to A, while the said semicircle makes one uniform revolution round the point C as a center.*

Put the radius EC $= a$, arch CRB $= v$, ordinate CB $= y$, and arch DB $= x$. Then (*art.* 41.) $\dot{x} = \frac{4y\dot{y}}{4\,\overline{a^2 - y^2}|^{\frac{1}{2}}}$; which, multiplied by $\frac{1}{2}y$, makes $\frac{1}{2}y\dot{x}$ (viz. the Fluxion of the Area, *art.* 114; for any space generated by the arch CRB is equal to that

* This curve was invented *Anno* 1756.

described by the ordinate CB;) $= \frac{2y^2\dot{y}}{\overline{4a^2-y^2}|^{\frac{1}{2}}} = \frac{2y^3\dot{y}}{\overline{4a^2y^2-y^4}|^{\frac{1}{2}}} = \frac{-4a^2y\dot{y}+2y^3\dot{y}}{\overline{4a^2y^2-y^4}|^{\frac{1}{2}}} + \frac{4a^2\dot{y}}{\overline{4a^2-y^2}|^{\frac{1}{2}}}$ = (because, *art.* 41. $\dot{v} = \frac{2a\dot{y}}{\overline{4a^2-y^2}|^{\frac{1}{2}}}$,) $- \overline{4a^2y^2-y^4}|^{-\frac{1}{2}} \times \overline{4a^2y\dot{y}-2y^3\dot{y}} + 2a\dot{v}$; the Fluent of which (*art.* 114. 2°.) is $-\overline{4a^2y^2-y^4}|^{\frac{1}{2}} + 2av = 2av - y \times \overline{4a^2-y^2}|^{\frac{1}{2}}$ = GC × CRB − CB × BG = four times the Area of the segment CRBC = the Area required. And, therefore, when the point B arrives at G, that is, when $y = 2a$, the Area of the whole spiral Space CHBLADC will be equal to four times the Area of the generating Semicircle.

CHAP. V.

Of finding the Convex Superficies of Solids.

Let the solid AIV be conceived to be generated by an enlarging or variable circle (whose increasing radius is the variable ordinate of the curve AI,) *Fig.* 71.

moving with a parallel motion along the axis from A to E: then will the velocity with which it's convex ſuperficies flows be equal to the periphery of the generating circle drawn into the velocity with which it moves along the curve AI; that is, the Fluxion of the ſaid ſuperficies, at any term HB, will be equal to the periphery of a circle, whoſe radius is the ordinate CB, drawn into the Fluxion of the curve at the point B.——This follows from *art.* 124. by conſidering the convex ſuperficies as always equal to the area of a curvilineal figure whoſe abſciſs is equal to the curve AH, and the ordinate as equal to the circumference of the generating circle BH whoſe radius is CB.

131. Hence, if we put the abſciſs AC $= x$, ordinate CB $= y$, curve AB $= z$, and $c = 6.28318$ &c. $=$ the circumference of a circle whoſe radius is 1 (*art.* 122.); then, becauſe $cy =$ the circumference of a circle whoſe radius is the ordinate CB, and (*art.* 118.) $\dot{z} = \overline{\dot{x}^2 + \dot{y}^2}|^{\frac{1}{2}}$; the General Expreſſion for the Fluxion of the Convex Superficies of any Solid ABH will be $= cy\dot{z}$ or $cy \times \overline{\dot{x}^2 + \dot{y}^2}|^{\frac{1}{2}}$; out of which, by help of the Equation of the given Curve AB, $\dot{x}^2$ or $\dot{y}^2$, &c. may be exterminated; and then, the Fluent of the reſulting expreſſion will give the Convex Snperficies required: As in the following Examples.

Example I.

132. To find the Superficies of a *Sphere*, or the Convex Superficies of any Segment of it.

Put the radius EA or EB $= a$, AC $= x$, CB $= y$, and AB $= z$. Let Bn = Cc expreſs the naſcent or very firſt Moment of AC, nb of CB, and Bb of AB; that is, let B$n = x'$, $nb = y'$, and B$b = z'$; then, Bb being conſidered as a little right line coinciding with a tangent to the point B, the triangles ECB and bnB will be alike: (for, $\angle$ CBn = $\angle$ EBb = a right angle; therefore, $\angle$ EBn being common, $\angle$ CBE = $\angle$ nBb; and, the angles at C and n being right, the angles CEB and Bbn muſt be likewiſe equal; *ergo*, &c.) Wherefore, by 4 E. 6. EB : BC :: bB : Bn, that is, $a : y :: z' : x'$; $\because z' = \frac{ax'}{y}$, or, (*art.* 7.) $\dot{z} = \frac{a\dot{x}}{y}$; which, ſubſtituted for $\dot{z}$, makes the general expreſſion for the Fluxion of the Convex Superficies, viz. $cy\dot{z}$ (*art.* 131.) $= cy \times \frac{a\dot{x}}{y} = ca\dot{x}$; the Fluent of which is cax = the Convex Superficies of the Segment ABH: And therefore, if for x be ſubſtituted $2a$, we ſhall have $2ca^2$ = the Superficies of the whole Sphere. *Fig.* 72.

Corollaries.

1. The Convex Superficies of any Segment of a Sphere is equal to the periphery of a great circle of that Sphere multiplied into the altitude of the Segment.

2. The whole Surface or Superficies of any Sphere is equal to the periphery of it's greateſt circle multiplied into it's diameter, or, equal to the Convex Superficies of it's circumſcribing Cylinder.

EXAMPLE II.

Fig. 73. 133. To find the Convex Superficies of the *Right Cone* AIV, whose Altitude AE is given $= a$ and base-diameter IV $= b$.

Put AC $= x$, and CB $= y$. Now, the triangles AEI and ACB being alike, by 4 E. 6. AE : EI :: AC : CB; that is, $a : \frac{1}{2}b :: x : y$; $\because x = \frac{2ay}{b}$; the Fluxion of which equation is $\dot{x} = \frac{2a\dot{y}}{b}$; $\because \dot{x}^2 = \frac{4a^2\dot{y}^2}{b^2}$; which, substituted for $\dot{x}^2$, makes the general expression for the Fluxion of the Convex Superficies, viz. $cy \times \overline{\dot{x}^2 + \dot{y}^2}|^{\frac{1}{2}}$ (*art.* 131.) $= cy \times \overline{\frac{4a^2\dot{y}^2}{b^2} + \dot{y}^2}|^{\frac{1}{2}} = \frac{c}{b} \times \overline{4a^2 + b^2}|^{\frac{1}{2}} y\dot{y}$; the Fluent of which is $\frac{c}{2b} \times \overline{4a^2 + b^2}|^{\frac{1}{2}} y^2 = \frac{cy^2}{b} \times \overline{a^2 + \frac{1}{4}b^2}|^{\frac{1}{2}} =$ (because by 47 E. 1. AI $= \overline{AE^2 + EI^2}|^{\frac{1}{2}} = \overline{a^2 + \frac{1}{4}b^2}|^{\frac{1}{2}}$,) $\frac{cy^2}{b} \times$ AI $=$ the Convex Superficies of the Segment ABH: And by writing $\frac{1}{2}b$ for y we have $\frac{1}{4}cb \times$ IA $=$ the Convex Superficies of the Cone AIV.

Corollary.

The Convex Superficies of any Right Cone is equal to half the circumference of it's base multiplied into it's slant height.

Example III.

134. To find the Convex Superficies of the *Parabolic Conoid* ABH.

Put the parameter of the parabola $= a$, AC $= x$ and CB $= y$: then, by the nature of the curve, $ax = y^2$; the Fluxion of which equation is $a\dot{x} = 2y\dot{y}$; Fig. 74.
therefore, $\dot{x} = \frac{2y\dot{y}}{a}$, and $\dot{x}^2 = \frac{4y^2\dot{y}^2}{a^2}$; which, substituted for $\dot{x}^2$, makes the general expression for the Fluxion of the Convex Superficies, viz. $cy \times \overline{\dot{x}^2 + \dot{y}^2}|^{\frac{1}{2}}$ (*art.* 131.) $= cy \times \overline{\frac{4y^2\dot{y}^2}{a^2} + \dot{y}^2}|^{\frac{1}{2}} = \frac{c}{a} \times \overline{4y^2 + a^2}|^{\frac{1}{2}} y\dot{y}$; the Fluent of which (*art.* 114.) is $\frac{c}{12a} \times \overline{4y^2 + a^2}|^{\frac{3}{2}}$: But, at the vertex A, where y, vanishes or $y = 0$, this fluential expression becomes $= \frac{c}{12a} \times \overline{a^2}|^{\frac{3}{2}} = \frac{ca^2}{12}$; therefore, the Fluent corrected (*art.* 117.) is $= \frac{c}{12a} \times \overline{4y^2 + a^2}|^{\frac{3}{2}} - \frac{ca^2}{12} =$ the Convex Superficies of the Solid ABH required.

CHAP. VI.

Of finding the Contents of Solids.

135. Let bc be conceived indefinitely near and parallel to the variable ordinate BC; and Bn equal Fig. 75.

and parallel to Cc the Increment of the variable abſciſs AC. Now, becauſe the little parallelogram $BCcn$ is expreſſive of the Increment or Moment of the plane figure ABC, (*art.* 124.) therefore, if the ſolid AIV be conceived to be generated by the revolution of the curvilineal figure AEI round the axis AE, the indefinitely little cylinder generated by the ſaid little parallelogram will expreſs the Moment or Increment of the ſolid at the term BH; and, becauſe this Moment or Increment is equal to the area of the circle deſcribed by the ordinate CB drawn into the Increment of the abſciſs AC, therefore, (*art.* 7.) the Fluxion of the Solid, at the term BH, is equal to the area of a circle, whoſe radius is the ordinate CB, drawn into the Fluxion of the abſciſs AC.

Or, Let the cylinder LG be generated by the parallel motion of the circle LD along the line LN; and, at the ſame time, the ſolid AIV by that of the concentric and variable circle ZF, which, at the vertex A, is ſuppoſed indefinitely ſmall, and continually enlarges as it moves along the curve AI. Then, it is plain, the ſolid AIV will increaſe ſlower or flow with a leſs degree of velocity, than the cylinder LG, before the generating circles arrive at the term HB; and afterwards faſter, or with a greater degree of velocity: therefore, at the ſaid term, (where the two generating circles become equal, or their peripheries coincide with each other,) they will increaſe, or flow, with the ſame or an equal degree of velocity. But, it is evident the velocity with which the cylinder flows is equal to the area of it's generating circle drawn into the velocity with which it moves along the line LN; that is, the Fluxion of the cylinder, at the term

HB, is equal to the area of a circle, whose radius is CB, drawn into the Fluxion of the line LH or AC: Therefore, the Fluxion of the Solid AIV, at the term BH, is equal to the area of a circle, whose radius is the ordinate CB, drawn into the Fluxion of the absciss AC; as before.

136. Hence, if we put the absciss AC $= x$, ordinate CB $= y$, and $c = 3.14159$ &c. = the semicircumference of a circle whose radius is 1 (*art.* 122.) then, the General Expression for the Fluxion of the Solid Content will be $= cy^2\dot{x}$: out of which, by help of the Equation of the given Curve, $\dot{x}$ or y^2 may be exterminated; and then, by finding the Fluent of the resulting fluxional expression, we shall have the Content of the Solid ABH required.

Example I.

137. To find the Content of a *Sphere*, or of any *Segment* of it.

Put AC $= x$, CB $= y$, and the diameter AD $= a$; then, by 35 E. 3. AC $\times$ CD $=$ BC $\times$ CH, that is, $ax - x^2 = y^2$; therefore, by writing $ax - x^2$ for y^2, the general expression for the Fluxion of the Solid Content viz. $cy^2\dot{x}$ (*art.* 136.) becomes $= c\dot{x} \times \overline{ax - x^2} = cax\dot{x} - cx^2\dot{x}$; the Fluent of which is $\frac{cax^2}{2} - \frac{cx^3}{3} = \frac{3cax^2 - 2cx^3}{6}$ = the Content of the Segment ABH: And therefore, if a be substituted Fig. 76.

for x, we shall have $\frac{3ca^3 - 2ca^3}{6} = \frac{1}{6}ca^3$ = the Content of the whole Sphere ABDH.

Hence, because four times the area of a great circle of the sphere is $= ca^2$, and the content of a cylinder circumscribing the sphere is $= \frac{1}{4}ca^3$, we have the following

Corollary.

The Content of any Sphere is equal to four times the area of it's greatest circle multiplied into $\frac{1}{6}$th part of it's axis, or, equal to two-third-parts of it's circumscribing cylinder.

EXAMPLE II.

Fig. 74. 138. To find the Content of the *Parabolic Conoid* ABH, generated by the parabolic space ABC revolving round the axis AC.

Put the parameter $= a$, AC $= x$, and CB $= y$; then, by the nature of the curve, $ax = y^2$. Now, by substituting ax for y^2, we have the general expression for the Fluxion of the Solid Content, viz. $cy^2\dot{x}$ (*art.* 136.) $= cax\dot{x}$; the Fluent of which is $\frac{1}{2}cax^2$ = (by writing y^2 for ax,) $\frac{1}{2}cxy^2$ = the Content of the Parabolic Conoid ABH required.

Or, the Fluxion of the equation of the curve, viz. of $ax = y^2$, is $a\dot{x} = 2y\dot{y}$; therefore $\dot{x} = \frac{2y\dot{y}}{a}$;

which, ſubſtituted for $\dot{x}$, makes the general expreſſion for the Fluxion of the Solid Content, viz. $cy^2\dot{x}$ (*art.* 136.) $= cy^2 \times \frac{2y\dot{y}}{a} = \frac{2cy^3\dot{y}}{a}$; the Fluent of which is $\frac{cy^4}{2a} =$ (by writing ax for y^2,) $\frac{1}{2}cxy^2 =$ the Solid Content; as before.

Corollary.

The Content of any Parabolic Conoid is equal to half of it's circumſcribing Cylinder.

Example III.

139. To find the Content of *any Cone* AIV whoſe baſe is a circle. *Fig.* 77.

Put the given altitude AE $= a$, and diameter VI $= b$. Let HB be parallel to VI; and put AC $= x$, and $c = .78539$ &c. $=$ the area of a circle whoſe diameter is 1. Now, the triangles AVI and AHB being ſimilar, by 4 E. 6. AE : VI :: AC : HB, that is, $a : b :: x : \frac{bx}{a} =$ HB; therefore, by 2 E. 12. the area of the circle HB is $= \overline{\frac{bx}{a}}\big|^2 \times c = \frac{cb^2x^2}{a^2}$; which (*art.* 135.) drawn into $\dot{x}$ is $\frac{cb^2x^2\dot{x}}{a^2} =$

the Fluxion of the Content of the Cone at the term CB; the Fluent of which is $\frac{cb^2 x^3}{3a^2}$ = the Content of the Cone AHB: And, therefore, by ſubſtituting a for x, we have $\frac{cb^2 a^3}{3a^2} = \frac{1}{3} cb^2 \cdot a$ = the Content of the Cone AVI required.

Or. Put the altitude AE $= a$. the area of the baſe VI $= b$, and AC $= x$: then, (becauſe ſimilar plane Figures are as the ſquares of their homologous ſides,) we ſhall have $a^2 : b :: x^2 : \frac{bx^2}{a^2} =$ the area of the ſection HB; which (*art.* 135.) drawn into $\dot{x}$ is $\frac{bx^2 \dot{x}}{a^2}$ = the Fluxion of the Cone AHB; the Fluent of which expreſſion gives the Content of the ſaid Cone $= \frac{1}{3} \times \frac{bx^3}{a^2}$: And therefore, by writing a for x, we have the Content of the whole Cone AVI $= \frac{1}{3} ab$. Hence, becauſe b may here ſtand for the area of the baſe of any Pyramid whatever, we have the following

Corollary.

The Solid Content of any Cone, or Pyramid, is equal to the area of it's baſe multiplied into one-third-part of it's perpendicular altitude; that is, it is equal to one-third part of a Cylinder, or Priſm, of the ſame altitude and baſe.

Example IV.

140. To find the Content of the Solid ABH, generated by the *Cissoidal* Space ABC revolving round the axis AE.* *Fig.* 78.

Put the axis AE $= a$; abscifs AC $= x$, and ordinate CB $= y$; then, (as was found in *art.* 34.) $x^3 = ay^2 - xy^2$: and therefore, $y^2 = \frac{x^3}{a - x}$; which substituted for y^2, makes the general expression for the Fluxion of the Solid Content, viz. $cy^2\dot{x}$ (*art.* 136.) $= \frac{cx^3\dot{x}}{a-x} = \frac{-cx^3\dot{x}}{x-a} = -cx^2\dot{x} - cax\dot{x} - ca^2\dot{x} - \frac{ca^3\dot{x}}{x-a} = -cx^2\dot{x} - cax\dot{x} - ca^2\dot{x} - ca^3 \times \frac{-\dot{x}}{a-x}$; the Fluent of which, (because *art.* 21. the Fluxion of the Hyp. Log. of $a - x$ is $\frac{-\dot{x}}{a-x}$,) is $= -\frac{cx^3}{3} - \frac{cax^2}{2} - ca^2x - ca^3 \times$ Hyp. Log. of $a - x$.

But, when $x = 0$, (as at A,) this Fluent becomes $= -ca^3 \times$ Hyp. Log. of a: therefore, the Fluent corrected (*art.* 117.) is $= -\frac{cx^3}{3} - \frac{cax^2}{2} - ca^2x - ca^3 \times$ Hyp. Log. of $\overline{a-x} + ca^3 \times$ Hyp.

* See how a *Cissoid* is generated, *art.* 34. *note.*

Log. of $a =$ (becauſe the Difference of the Logarithms of any two numbers is equal to the Logarithm of their Quotient,) $- cx \times \frac{2x^2 + 3ax + 6a^2}{6}$ $+ ca^3 \times$ Hyp. Log. of $\frac{a}{a - x} =$ the Content of the Solid ABH required.

Corollary.

When $x = \frac{1}{2}a$, the Content of the Solid will be $= ca^3 \times : -\frac{2}{3} +$ Hyp. Log. of $2 = ca^3 \times 0.02648$ &c. (See *Part* III. *Queſt.* 9.)

PART III.

MISCELLANEOUS QUESTIONS,

With their *Incremental* and *Fluxional* SOLUTIONS.

I.

In an *Ellipsis* ABD, whose Foci are the points F and K, if a right line Bt be drawn bisecting the angle FBK; then will the said line be perpendicular to the tangent TBG. *Quære* the Demonstration. *Fig.* 79.

Let the point b be supposed indefinitely near to B; and with the lines FB and Kb, as radii, describe the arches Bm and bn: then, if we consider the said arches as little right lines perpendicular to Fb and KB respectively, (because the Sum of the lines FB and KB is always the same invariable quantity, viz. $=$ AD, and therefore the Increment mb $=$ the Decrement nB,) the right angled triangles B$n\odot$ and $bm\odot$ will be equal and similar, as will therefore the right angled triangles Bmr and bnr; and therefore, if Bb, the Increment of the

curve AB, be ſuppoſed to coincide with the tangent BG, the angles rbB and rBb will be equal, that is, $\angle$ FBT $=$ $\angle$ KBG. Therefore, the right line BT, which biſects the $\angle$ FBK, makes the $\angle$ tBT $=$ $\angle$ tBG $=$ a Right angle. Q. E. D.

II.

Fig. 80. In an *Hyperbola* AB, whoſe Focus is the point F, and tranſverſe Diameter is DA $=$ KA $-$ AF; the right line tB, which biſects the angle KBF, is a tangent to the curve at the point B. *Quære* the Demonſtration.

Suppoſe the point b indefinitely near to B, and deſcribe the little arches Bm and Bn with the radii KB and FB. Now, becauſe the Difference of the lines KB and FB is always the ſame invariable quantity, viz. $=$ DA, the Increments mb and nb are equal; and therefore, if the arches Bm and Bn be conſidered as little right lines perpendicular to Kb and Fb reſpectively, and Bb (the Increment of the curve AB) as coinciding with the tangent; the little right angled triangles Bmb and Bnb will be equal and ſimilar. Therefore, the right line tB, biſecting the angle KBF, is a tangent to the curve at the point B. Q. E. D.

III.

Fig. 81. In a *Parabola* AB, whoſe Focus is the point F, if a right line KB be drawn parallel to the axis, and the angle KBF be biſected by the right line

tB; then will this line be a tangent to the curve at the point B. *Quære* the Demonſtration?

Let the indefinite right line LK be perpendicular to LF, and LA = AF; draw kb indefinitely near and parallel to KB, and Bm equal and parallel to Kk; and with FB, as a radius, deſcribe the little arch Bn. Now, becauſe the lines KB and FB are always equal to each other, the Increment mb is = the Increment nb; and therefore, (the indefinitely ſmall arch Bn being conſidered as a little right line perpendicular to Fb, and the Increment of the curve, Bb, as coinciding with the tangent,) the little right angled triangles Bmb and Bnb are equal and ſimilar. Conſequently, the right line tB, which biſects the angle KBF, is a tangent to the curve at the point B. Q. E. D.

IV.

Quære the Nature of the Curve APB?—ſuppoſing FP or CE, the diſtance of the parallel and indefinite right lines AC and PE, to be given; the right line AD to paſs through the point P; and CE × CD = CB².* *Fig.* 82.

Put CE = a, abſciſs AC = x, ordinate CB = y and CD = z. Let cd be ſuppoſed indefinitely

* To describe the Curve, or to find the Point B in the line CD through which it must pass —Produce CD to G, making DG = FP; describe the semicircle GHC, and draw the perpendicular ordinate DH; lastly, make CB = DH: then will B be the Point required. For, by 35 E. 3. GD × DC = DH²; that is, EC × CD = CB².

near and parallel to CD; and Dm, Bn, equal and parallel to Cc: and put $nb = y'$, $md = z'$; and suppose TB a tangent to the curve at the point B. Now, by 4 E. 6. DC : CA :: $dm : mD$, that is, $z : x :: z' : \frac{xz'}{z} = Dm$ or Bn; and $bn : nB$ :: BC : CT, that is, $y' : \frac{xz'}{z} :: y : \frac{xyz'}{zy'} =$ CT, or (*art.* 7.) $\frac{xy\dot{z}}{z\dot{y}} =$ CT. But, by the question given, $az = y^2$; the Fluxion of which equation is $a\dot{z} = 2y\dot{y}$ ∵ $\dot{z} = \frac{2y\dot{y}}{a}$; which, substituted for $\dot{z}$, makes the above value of the Subtangent CT (viz. $\frac{xy\dot{z}}{z\dot{y}}$) $= \frac{2xy^2}{az} =$ (because $\frac{y^2}{az} = 1$,) $2x$. Therefore, (*art.* 28.) the curve AB is a *Parabola*, whose vertex is A.

Corollary.

If F be the Focus; then, by the nature of the Parabola, PF $=$ 2FA; and therefore DC $=$ 2CA $=$ CT, that is, $z = 2x$.

V.

Fig. 83. If TB be a Tangent to the given Curve AB; and another Curve AD be so described as that it's ordinate CD shall always be in a given ratio to the corresponding Segment of the former Curve:

then, BT will be to TC as the correſponding Segment of the Curve AB is to the Subtangent CV. *Quære* the Demonſtration?

Put $CT = s$, $TB = t$, $CD = y$, $AB = z$: and let cd be ſuppoſed indefinitely near and parallel to CD; and Dm, Bn, equal and parallel to Cc; that is, let $md = y'$, and $Bb = z'$. Now, $BT : TC :: bB : Bn$, that is, $t : s :: z' : \frac{sz'}{t} = Bn$ or Dm; and $dm : mD :: DC : CV$, that is, $y' : \frac{sz'}{t} :: y : \frac{syz'}{ty'} = CV$, or (*art.* 7.) $\frac{sy\dot{z}}{t\dot{y}} = CV$. Let the given ratio of DC to AB be as a to b, that is, $y : z :: a : b$, $\because z = \frac{by}{a}$; the Fluxion of which equation is $\dot{z} = \frac{b\dot{y}}{a}$; which ſubſtituted for $\dot{z}$ makes the above $\frac{sy\dot{z}}{t\dot{y}} = CV = \frac{bsy}{at}$ (which, by writing z for it's value $\frac{by}{a}$, is) $= \frac{sz}{t}$; therefore, $t : s :: z : CV$, that is, $BT : TC :: AB : CV$. Q. E. D.

VI.

In the Curve ABD, whoſe Equation (putting the abſciſs $AC = x$, ordinate $CB = y$, and the baſe $AD = a$,) is $ax - x^2 = ay + y^2$: required *Fig.* 84.

M

the Radius of Curvature for any point, and a geometrical Conſtruction to illuſtrate and confirm the Work.

The Fluxion of the given equation of the curve is $a\dot{x} - 2x\dot{x} = a\dot{y} + 2y\dot{y}$; which, making $\dot{x} = 1$, is $a - 2x = a\dot{y} + 2y\dot{y}$; and the Fluxion of this equation again, the Fluxion of $\dot{y}$ being conſidered as negative, is $-2 = -a\ddot{y} + 2\dot{y}^2 - 2y\ddot{y}$. Hence we have $\dot{y} = \dfrac{a - 2x}{a + 2y}$, $\dot{y}^2 = \dfrac{a^2 - 4ax + 4x^2}{a^2 + 4ay + 4y^2}$,

and $\ddot{y} = \dfrac{4a^2 + 8ay + 8y^2 - 8ax + 8x^2}{\overline{a + 2y}|^3} =$ (becauſe by the equation of the curve, $8ay + 8y^2 = 8ax - 8x^2$, or $8ay + 8y^2 - 8ax + 8x^2 = 0$,) $\dfrac{4a^2}{\overline{a + 2y}|^3}$. Now, by ſubſtituting for $\dot{y}^2$ and $\ddot{y}$ theſe their values, in the general expreſſion for the Radius of Curvature, which was found in *art.* 74. to be $=$ $\dfrac{\overline{1 + \dot{y}^2}|^{\frac{3}{2}}}{\ddot{y}}$ when $\dot{x} = 1$ and the Fluxion of $\dot{y}$ is negative, we ſhall have $\overline{\dfrac{2a^2 + 4ay + 4y^2 - 4ax + 4x^2}{\overline{a + 2y}|^2}}\Big|^{\frac{3}{2}}$

$\times \dfrac{\overline{a + 2y}|^3}{4a^2} =$ (becauſe $4ay + 4y^2 - 4ax + 4x^2$

$= 0$,) $\overline{\dfrac{2a^2}{\overline{a + 2y}|^2}}\Big|^{\frac{3}{2}} \times \dfrac{\overline{a + 2y}|^3}{4a^2} = \dfrac{\overline{2a^2}|^{\frac{3}{2}}}{4a^2} = \dfrac{\overline{8}|^{\frac{1}{2}}a}{4}$

$= \overline{\tfrac{1}{2}}|^{\frac{1}{2}} a =$ the Radius of Curvature required;

which being a fixed or invariable quantity proves the curve to be an *Arch* of a *Circle.*

Now, if the Radius of a Circle be $\overline{\tfrac{1}{2}}\rceil^{\frac{1}{2}} a$; then, a is the ſide of it's inſcribed ſquare, or the chord of 90°. as is the right line AD: For, if the ſaid right line AD be biſected in F, and the perpendicular FE be drawn $=$ AF $= \frac{1}{2} a$; the point E will be the center of the circle; and conſequently, the radius will be AE ($= \overline{EF^2 + FA^2}\rceil^{\frac{1}{2}}$) $= \overline{\tfrac{1}{2} a^2}\rceil^{\frac{1}{2}} = \overline{\tfrac{1}{2}}\rceil^{\frac{1}{2}} a$. And, that x, in the given equation, muſt flow in the ſaid chord AD, may be thus demonſtrated: Draw the right line BC perpendicular to AD, and let it be produced until it meets the circle's periphery in G; and let HI be drawn equal and parallel to AD: then, it is evident, that, CK $=$ AH $=$ AD, by conſtruction; and KG $=$ CB, becauſe AC $=$ HK, and AB $=$ HG; therefore, CG $=$ AD $+$ CB: But, by 35 E. 3. AC $\times$ CD $=$ BC $\times$ CG, that is, (putting AD $= a$, AC $= x$, and CB $= y$,) $x \times \overline{a - x} = y \times \overline{a + y}$, or $ax - x^2 = ay + y^2$. Therefore, &c.

SCHOLIUM.

From the Conſtruction here given, it appears, that the Queſtion may be diverſified ſo as to be adapted to any regular Polygon that can be inſcribed in a Circle. We ſee here, alſo, a demonſtration of the juſtneſs of the *fluxional* Calculus as made uſe of above.

VII.

Suppose the Earth to revolve in a circular orbit round the Sun as it's center; and the Moon to revolve round the Earth in the same manner; as also, that the planes of their orbits do coincide; and that the diameters of the said orbits are as 340 to 1; and lastly, that the Moon performs 13.368 revolutions to every single revolution of the Earth. *Quære* the Nature and Description of the Curve generated by the Center of the Moon; or, whether the Curve described by the Center of the Moon, in one Lunation, be any-where *Convex* towards the Sun?

Fig. 85. Let S represent the Sun; E, the Earth; Ee, an arch of the orbit of the Earth passed over by it's center in one lunation of the Moon; the circumference of the circle EAF = the concentric arch Aα: Then, (because $13.368 - 1 = 12.368$ = the number of lunations in a year or one revolution of the Earth, and therefore SA : EA :: 12.368 : 1,) when the Moon is in conjunction with the Sun, the distance between the Sun and Moon will be greater than the distance or radius SA. Now, the Curve described by the center of the Moon is the same as that described by a point M (EM being the semidiameter of the Moon's orbit,) carried round by the rotation of the circle EAF on the arch Aα: It is therefore of the *Cycloidal* Kind, having a point of Inflection, if *every* Cycloid described by a point within the generating circle is inflected as well upon a *circular* as upon a *rectilinear* base (*art.* 65.). To determine which,

Put SA or SR $= a$, EA or eR $= b$, EM or em $= c$, Rm $= r$, R$d = s$; and let mC be the Radius of Curvature at any point m, which, it is evident, muſt paſs through the point of contact R. Suppoſe the point n indefinitely near to m: Then, Rr and Rr being the indefinitely ſmall contemporary arches with mn, and conſequently the triangles Rmr and Rnr equal in all reſpects; if we conſider the ſaid little arches Rr and Rr as little right lines perpendicular to the radii er and Sr, we ſhall have the $\angle\, m\mathrm{R}n = \angle\, r\mathrm{R}r =$ (becauſe the angles $e\mathrm{R}r$ and S$\mathrm{R}r$ added to either ſide of the equation make it two right angles,) $\angle\, \mathrm{R}er + \angle\, \mathrm{RS}r$. Now, SR : eR :: $\angle\, \mathrm{R}er : \angle\, \mathrm{RS}r$, and SR : SR + R$e$ :: $\angle\, \mathrm{R}er : \angle\, \mathrm{R}er + \angle\, \mathrm{RS}r$, that is, $a : a + b :: \angle\, \mathrm{R}er : \angle\, m\mathrm{R}n = \frac{a + b}{a} \angle\, \mathrm{R}er$. Again, in any triangle, as dmr, if the angles mdr, mrd, and Rmr the complement of the obtuſe angle to two right angles, be indefinitely ſmall, they will be proportional to the oppoſite ſides mr, md, and dr*; that is, $dr : md :: \angle\, \mathrm{R}mr : \angle\, mrd$; and $dr - md : dr :: \angle\, \mathrm{R}mr - \angle\, mrd : \angle\, \mathrm{R}mr$, that is, $m\mathrm{R} : d\mathrm{R}$ ::

* For, let the triangle be circumſcribed (as in *fig.* 86.) by the circle rmd: then will the arches dm and mr differ infinitely little from their chords dm and mr, which therefore may be taken as equal to them. And, ſince by 20 E. 3. the angle at the center of a circle is double of the angle at the periphery, the arches or chords rm and md are the meaſures of double the angles mdr and mrd reſpectively; or, the angles mdr and mrd are to each other as their oppoſite ſides rm and md: and becauſe by 32 E. 1. the angle Rmr is equal to the ſum of the angles mdr and mrd; therefore the meaſure of it is the ſum of the arches or chords rm and md; which differing infinitely little from the ſide or chord rd, may be conſidered as equal to it. *Ergo*, &c. *Fig.* 86.

$\angle$ Rdr : $\angle$ Rnr, or, $r : s :: \frac{1}{2} \angle$ Rer : $\angle$ Rnr $=$ $\frac{s}{2r} \angle$ Rer. And again, $\angle$ RCn : $\angle$ RnC :: Rn : RC, that is, $\angle$ mRn — $\angle$ Rnr : $\angle$ Rnr :: Rm : RC, or, $\overline{\frac{a+b}{a} - \frac{s}{2r}} \angle$ Rer : $\frac{s}{2r}$ $\angle$ Rer :: r : RC $= \frac{ars}{2ar + 2br - as}$. Confequently, mR + RC $=$ mC $= \frac{2ar^2 + 2br^2}{2ar + 2br - as} = \frac{r^2}{r - \frac{as}{2a+2b}} =$ the Radius of Curvature at any point m.

Now, it is evident, that, at the point of Inflection, the Radius of Curvature muft be Infinite; or that, on one fide of the faid point, the expreffion for the radius of curvature muft be affirmative, and on the other negative; therefore, r muft be more than $\frac{as}{2a + 2b}$ on one fide of the faid point, and on the other lefs; and confequently, at the point of Inflection, $r = \frac{as}{2a + 2b}$; which fubftituted for r, makes ($dm \times m$R $=$) $rs - r^2 = \frac{2abs^2 + a^2 s^2}{\overline{2a + 2b}|^2} =$ (becaufe $dm \times m$R $= fm \times ma =$) $b^2 - c^2$; from which equation we have $s = \frac{\overline{a + 2b}\sqrt{b^2 - c^2}}{\sqrt{2ab + a^2}}$. —Or, to find r, fay $2ar + 2br = as$, or, $s =$

$\frac{2ar + 2br}{a}$; then, ($dm \times mR =$) $rs - r^2 =$ $\frac{ar^2 + 2br^2}{a} = (fm \times ma =)\ b^2 - c^2$; which equation gives $r = \sqrt{\frac{ab^2 - ac^2}{a + 2b}}$, when the point m becomes a point of Inflection.

Now, as mR (r) muſt, by the nature of the circle, always be greater than ma; that is, as $\sqrt{\frac{ab^2 - ac^2}{a + 2b}}$ muſt always be more than $b - c$; and conſequently, $\frac{ab^2 - ac^2}{a + 2b}$ be more than $\overline{b - c}|^2$, that is, $\frac{ab + ac}{a + 2b} \times \overline{b - c}$ be more than $\overline{b - c} \times \overline{b - c}$; therefore, c muſt always be more than $\frac{b^2}{a + b}$; that is EM muſt be *more* than a third proportional to ES and EA in order to have a point of Inflection take place in the Curve: But, in the preſent caſe, ES, EA, and EM, being as 13.368,1, and $\frac{13.368}{340}$ or 039; therefore, EM is *less* than the ſaid third proportional; and conſequently, the Curve M*mu*, generated by the Center of the Moon, has *not* a point of Inflection, or, is *no-where* Convex towards the Sun. Q. E. I.

Corollaries.

1. When $r = s$, that is, when the point m coincides with a, or a is the generating point; then, the Radius of Curvature will be $= \frac{2ar^2 + 2br^2}{ar + 2br}$ $= \frac{a + b}{a + 2b} 2r$; and RC $= \frac{a}{a+2b} r$; by analogy, $a + 2b : a :: r :$ RC, that is, SF : SA :: mR : RC.

2. When a is Infinite, and $r = s$; that is, when the base becomes a right line, and the point m coincides with a, or the curve is the common Cycloid; the Radius of Curvature will be $= 2r$. For then, $2br^2$ and $2br$ will be infinitely little in comparison of $2ar^2$ and ar, and therefore may be rejected.

Fig. 87. 3. When a is Infinite, that is, when the base degenerates into a right line, or the curve is the protracted or interior Cycloid, as in *fig.* 87. the Radius of Curvature will become $= \frac{2ar^2}{2ar - as} = \frac{2r^2}{2r - s}$; and, therefore, at the point of Inflection, where the radius of curvature is infinite, $2r = s$, that is, Rm $= md$; and consequently the right line Rd is perpendicular to the radius ea, and the point a is in the base NM. Whence, to find the point of Inflection we have the following Construction, viz. Make PR $= ag$, or AR $=$ Pg; draw Re equal and parallel to the radius PO; make $ra =$ Ng; draw the right line ea; and, lastly, make $em =$ ON: Then will

m be the point of Inflection in the interior Semicycloid μmM.

VIII.

The *Fluxion* of the Hyperbolic Logarithm of any quantity, is equal to the Fluxion of that quantity divided by the quantity itself. *Quære* the Demonstration? (See *art.* 21.)

Let YAI be an Hyperbola, whose assymptotes are the perpendicular right lines EZ and ET, and whose parameter is AP = EP = 1; draw any ordinate CB parallel to PA: then, (as Writers on *Conics* demonstrate,) the Space PABC will be the Hyperbolic Logarithm of the line EC; and, therefore, the Fluxion of the space PABC will be equal to the Fluxion of the Hyperbolic Logarithm of the line EC. Now, the Fluxion of this space, putting EC $= x$ and CB $= y$, is (by *art.* 124.) $= y\dot{x}$; and, by the known property of the curve, EC : EP :: PA : CB, that is, $x : 1 :: 1 : y = \frac{1}{x}$: therefore $y\dot{x} = \frac{\dot{x}}{x}$ that is, the Fluxion of the space PABC, or, of the Hyperbolic Logarithm of the line EC, is equal to the Fluxion of the said line, divided by the line itself. Q. E. D. *Fig.* 88.

N. B. By a Space or Line, is meant it's Numerical Measure.

IX.

The Hyperbolic Logarithm of 1 being 0, what is the Hyperbolic Logarithm of 10; (See *art.* 21.)

Fig. 89. Let EZ and ET be the aſymptotes of the rectangular Hyperbola YAI, whoſe parameter is AP = PE = 1. Then, the area of the ſpace PABC will be the Hyperbolic Logarithm of $\frac{EC}{EP}$, that is, of EC; the area of the ſpace PA*bc* will be the Hyperbolic Logarithm of $\frac{EP}{Ec}$, that is, of $\frac{1}{Ec}$; and, the area of the ſpace *cb*BC will be the Hyperbolic Logarithm of $\frac{EC}{Ec}$; the right lines or ordinates CB and *cb* being ſuppoſed parallel to the aſymptote EZ. Now, to find theſe areas.

1°. Put P*c* = *x*, and *cb* = *y*; then, by the known property of the curve, $y = \frac{1}{1-x}$; which, (*art.* 124.) drawn into $\dot{x}$, is $y\dot{x} = \frac{\dot{x}}{1-x}$ = the Fluxion of P*b* = (by throwing $\frac{\dot{x}}{1-x}$ into a Series, *art.* 102.) $\dot{x} + x\dot{x} + x^2\dot{x} + x^3\dot{x} + x^4\dot{x} + x^5\dot{x} + x^6\dot{x} + x^7\dot{x} + x^8\dot{x} + x^9\dot{x} + x^{10}\dot{x} + x^{11}\dot{x}$ + &c. therefore the Fluent of this ſeries, viz. $x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{5}x^5 + \frac{1}{6}x^6 + \frac{1}{7}x^7 + \frac{1}{8}x^8 + \frac{1}{9}x^9 + \frac{1}{10}x^{10} + \frac{1}{11}x^{11} + \frac{1}{12}x^{12}$ + &c. is = P*b*.

2° Put PC = *x*, and CB = *y*; then, $y = \frac{1}{1+x}$, and (*art.* 124.) $y\dot{x} = \frac{\dot{x}}{1+x}$ = the Fluxion of PB

$=$ (by throwing $\frac{\dot{x}}{1+x}$ into a Series, *art.* 103.) $\dot{x} - x\dot{x} + x^2\dot{x} - x^3\dot{x} + x^4\dot{x} - x^5\dot{x} + x^6\dot{x} - x^7\dot{x} + x^8\dot{x} - x^9\dot{x} + x^{10}\dot{x} - x^{11}\dot{x} +$ &c. therefore the Fluent of this series, viz. $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \frac{1}{6}x^6 + \frac{1}{7}x^7 - \frac{1}{8}x^8 + \frac{1}{9}x^9 - \frac{1}{10}x^{10} + \frac{1}{11}x^{11} - \frac{1}{12}x^{12} +$ &c. is $=$ PB.

Hence,

If P*c* $=$ PC, the area *cb*BC, viz. the ſum of P*b* and PB, will be $= 2 \times : x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 + \frac{1}{9}x^9 + \frac{1}{11}x^{11} +$ &c. And P*b* — PB will be $= x^2 + \frac{1}{2}x^4 + \frac{1}{3}x^6 + \frac{1}{4}x^8 + \frac{1}{5}x^{10} + \frac{1}{6}x^{12} +$ &c. That is, if E*c* $=$ .9, and *c*P $=$ PC $=$ .1; then, by ſubſtituting .1 for x, we ſhall have *c*B $=$ .2006706954 &c. and P*b* — PB $=$ .0100503358 &c. half of which added to half *c*B is .1053605156 &c. $=$ P*b* $=$ the Hyp. Log. of $\frac{1}{.9}$. And, if E*c* $=$ .8, and *c*P $=$ PC $= x =$ .2; then, by writing .2 for x, we have *c*B $=$ .4054651081 &c. $=$ the Hyp. Log. of $\frac{1.2}{.8}$; and P*b* — PB $=$ .0408219945 &c. half of which ſubtracted from half *c*B is .1823215567 &c. $=$ PB $=$ the Hyp. Log. of $\frac{1.2}{1}$ or 1.2; and this ſubtracted from *c*B leaves .2231435513 &c. $=$ P*b* $=$ the Hyp. Log. of $\frac{1}{.8}$.

Therefore, by the Nature of Logarithms, the Hyp. Log. of $\frac{1}{.9} \times \frac{1.2}{.8} \times 1.2$, viz. of 2, is $= .1053605156$ &c. $+ .4054651081$ &c. $+ .1823215567$ &c. $=$ $.6931471805$ &c. and, the Hyp. Log. of 2^3, viz. of 8, is $= 3 \times .6931471805$ &c. $= 2.0794415416$ &c. and, the Hyp. Log. of $8 \times \frac{1}{.8}$, viz. of 10, is $= 2.0794415416$ &c. $+ .2231435513$ &c. $=$ 2.3025850929 &c. Q. E. I.

X.

Fig. 90. Let the given right line CA turn uniformly round the point C as a center; and, let a point be supposed to pass with an uniform motion, from A, along the right line AF, equal and perpendicular to the said line CA, and such velocity, as to arrive at F at the same time that the said lines come to be in their first situation: then, by this point, will the *Spiral* APBF be described. *Quære* the Area of any Space ARaBPA?*,

Put CA $=$ AF $= a$, the circumference of the circle ARaA $= b$, aB $= v$, arch DB (described with the ordinate or radius CB,) $= x$, CB $= y$, arch ARa $= z$, B$n = \dot{x}$, and $aa = \dot{z}$. Now, (supposing Bn, the Increment of the arch DB, to be a little right line perpendicular to the radius CB,) if we draw aG perpendicular to CB, the Moment or Increment of the space ARaBD, viz. aBnaa,

* This Curve was invented *Anno* 1756.

will by 41 E. 1. be $= \frac{1}{2}$ GB $\times$ Bn $=$ (becauſe by 8 and 4 E. 6. CB : Ba :: aB : BG, or, $y : v :: v :$ BG $= \frac{v^2}{y}$,) $\frac{v^2 \dot{x}}{2y}$; therefore, *art.* 7. the Fluxion of the ſaid ſpace, which is $=$ the Fluxion of the ſpace in queſtion, is $= \frac{v^2 \dot{x}}{2y}$. But, it is plain from the generation of the Curve, *(af)* $a : b :: \dot{v} : \dot{z} = \frac{b\dot{v}}{a}$, and (C$a$) $a : y :: \dot{z} : \dot{x} = \frac{y\dot{z}}{a} = \frac{y}{a} \times \frac{b\dot{v}}{a} = \frac{by\dot{v}}{a^2}$; which ſubſtituted for $\dot{x}$ makes the above Fluxion of the ſpace in queſtion $= \frac{bv^2 \dot{v}}{2a^2}$, the Fluent of which is $\frac{bv^3}{6a^2} =$ the Area of the Space required: And therefore, by writing a for v, we ſhall have the Area of the whole ſpiral Space ARAFBPA $= \frac{1}{6} ab = \frac{1}{3}$ the Area of the Circle CARaA. Q. E. I.

Or, By 47 E. 1. $v = \overline{y^2 - a^2}|^{\frac{1}{2}}$; the Fluxion of which equation is $\dot{v} = \frac{y\dot{y}}{\overline{y^2 - a^2}|^{\frac{1}{2}}}$; therefore $\dot{x}$ ($= \frac{by\dot{v}}{a^2}$) $= \frac{b}{a^2} \times \frac{y^2\dot{y}}{\overline{y^2 - a^2}|^{\frac{1}{2}}}$; which ſubſtituted for $\dot{x}$, makes $\frac{1}{2} y\dot{x}$ viz. the Fluxion of the Area *art.* 124. (for CB and CaB deſcribe equal Spaces,) $= \frac{b}{2a^2} \times \frac{y^3\dot{y}}{\overline{y^2 - a^2}|^{\frac{1}{2}}} = \frac{b}{6a^2} \times \frac{3y^5\dot{y} - 2a^2 y^3 \dot{y}}{\overline{y^6 - a^2 y^4}|^{\frac{1}{2}}} + \frac{1}{3} b \times$

$\frac{y\dot{y}}{\overline{y^2 - a^2}|^{\frac{1}{2}}}$; the Fluent of which expression is $\frac{b}{6a^2} \times \overline{y^6 - a^2 y^4}|^{\frac{1}{2}} + \frac{1}{3} b \times \overline{y^2 - a^2}|^{\frac{1}{2}}$ = the Area of the Space ACaBPA. So that, by writing $2a^2$ for y^2, we have the area of the Space AFBPA $= \frac{2}{3}ab$; from which if we take $\frac{1}{2}ab$ (the Area of the Circle ARA,) there will remain $\frac{1}{6}ab$ = the Area of the whole spiral Space; as before.

Corollary.

$\frac{x\dot{y}}{\dot{y}}$ (*art.* 38.) is $= \frac{y}{\dot{y}} \times \frac{by^2\dot{y}}{a^2 \times \overline{y^2 - a^2}|^{\frac{1}{2}}} = \frac{by^3}{a^2 \times \overline{y^2 - a^2}|^{\frac{1}{2}}} = \frac{by^3}{a^2 v}$ = the Subtangent CT.

Construction. Through the center C draw the indefinite right line AT perpendicular to the ordinate CB; describe through the points A and B a semicircle ABH; make CI = the circumference of the circle ARA, and CK = aB; draw right lines HK and IL, and parallel to IL draw MB; lastly, produce CB to N, making CN = CM, and draw a right line NT parallel to KH: then will T be the point from which a Tangent to the point B must be drawn. For, (AC) $a : y :: y : \frac{y^2}{a}$ = CH; and (LC) a : (CI) b :: (BC) y : CM $= \frac{by}{a}$ = CN; and (KC) v : (CH) $\frac{y^2}{a}$ :: (NC) $\frac{by}{a}$: CT $= \frac{by^3}{a^2 v}$.

XI.

Quære the Content of the cylindrical Ring ABDR? — the radius EA of the inner circumference being given $= a$; and AD, the diameter of the ring, or of the generating circle, $= b$. *Fig.* 91.

Put any arch LA $= x$, and .78539 &c. $= c$: suppose Ed indefinitely near to ED; and describe the concentric pricked circle CR, making AB $=$ BD $= \frac{1}{2} b$. Then, the Moment of the ring, viz. Ad, will be equal to the area of the generating circle AD drawn into the Increment Bb. Now, EA: Aa :: EB : Bb, that is, $a : x' :: a + \frac{1}{2} b :: x' + \frac{bx'}{2a}$ $=$ Bb : and the area of the circle AD $= b^2 c$: therefore, the Moment of the ring is $= b^2 c \times \overline{x' + \frac{bx'}{2a}} = b^2 cx' + \frac{b^3 cx'}{2a}$, or it's Fluxion (*art.* 7.) $=$ $b^2 c\dot{x} + \frac{b^3 c\dot{x}}{2a}$; the Fluent of which is $b^2 cx + \frac{b^3 cx}{2a}$ $= \overline{1 + \frac{b}{2a}} \times b^2 cx =$ the Content of the Ring from L to A; and therefore, by substituting $8ac$ for x, we have the content of the whole Ring $= \overline{1 + \frac{b}{2a}}$ $\times 8ab^2 c^2 = b^2 c \times \overline{2a + b} \times 4c =$ the area of the generating circle AD drawn into the circumference of the pricked circle BR. Q. E. I.

XII.

If a heavy Sphere, whose diameter is 4 inches, be let fall into a conical Glass $\frac{1}{5}$th full of Water, whose diameter is 5 inches and altitude 6; how much of the Sphere will be immersed in the Water?

Fig. 92. Put the altitude VH $= 6 = a$; radius HF $= 2.5 = b$; 3.14159 &c. (*art.* 122.) $= c$; AD, the diameter of the sphere, $= 4 = d$; and AC, that part of the said diameter under the water, $= x$; then, CD $= d - x$. Now, the capacity of the glass is $= \frac{1}{3} ab^2 c$; and therefore, by the question, the quantity of water in it is $= \frac{1}{15} ab^2 c$. By 35 E. 3. AC $\times$ CD $=$ CB^2, that is, $dx - x^2 =$ the square of the radius of the section of the sphere; therefore the area of the said section is $= cdx - cx^2$; which drawn into $\dot{x}$ is $cdx\dot{x} - cx^2\dot{x} =$ the Fluxion of the segment of the sphere under the water; the Fluent of which is $\frac{1}{2} cdx^2 - \frac{1}{3} cx^3 =$ the content of the said segment. Hence (similar solids being as the cubes of their homologous sides,) $\frac{1}{3} ab^2 c : a^3 :: \frac{1}{15} ab^2 c + \frac{1}{2} cdx^2 - \frac{1}{3} cx^3 : \frac{1}{5} a^3 + \frac{3a^2 dx^2}{2b^2} - \frac{a^2 x^3}{b^2} =$ VC^3. But, VF $= \sqrt{a^2 + b^2}$; and HF : FV :: GE : EV, that is, $b : \sqrt{a^2 + b^2} :: \frac{1}{2} d : \frac{d}{2b} \sqrt{a^2 + b^2} =$ EV; $\because$ VA ($=$ VE $-$ AE) $= \frac{d}{2b} \sqrt{a^2 + b^2} - \frac{1}{2} d = 3.2$, which put $= e$, then VC $= e + x$, and VC$^3 = e^3 + 3e^2 x + 3ex + x^3$

= (becauſe by the above VG^3 is =) $\frac{1}{3}a^3 + \frac{3a^2dx^2}{2b^2} - \frac{a^2x^3}{b^2}$. Now, this equation produces $\overline{10a^2+10b^2}\cdot x^3 + \overline{30b^2e - 15a^2d}.\, x^2 + 30b^2e^2x = 2a^3b^2 - 10b^2e^3$, that is, $422.5x^3 - 1560x^2 + 1920x = 652$; which equation divided by 422.5 is $x^3 - 3.692x^2 + 4.544x = 1.543$; whence x may be found = .546 = AC = that Part of the Diameter of the Sphere under the Water. Q. E. I.

SCHOLIUM.

WE might now proceed to the inveſtigation of the Centers of *gravity*, *percuſſion*, and *oscillation*, and a great variety of other Problems in the various branches of Mathematical and Philoſophical Science: but, this Tract being intended as an *Introduction* only, for theſe Things we muſt refer the Learner to the larger and more extenſive Books on the Subject*; in which, though he may meet with many Difficulties, it is hoped they are not ſuch but he will now be able to ſurmount.

The Books, in *Engliſh*, profeſſedly on the Subject, are,

1. A Treatiſe of Fluxions: or, an Introduction

* The Author particularly refers to the Works of his two celebrated Friends, Mr. *Emerson* and the late Mr. *Simpson*.

to Mathematical Philoſophy. Containing a full explication of that Method by which the moſt celebrated Geometers of the preſent Age have made ſuch vaſt Advances in Mechanical Philoſophy: By CHARLES HAYES, *Gent.*——Folio. 315 Pages. Cuts, 1704.

2. An Inſtitution of Fluxions: Containing the firſt Principles, the Operations, with ſome of the Uſes and Applications of that admirable Method. By HUMPHRY DITTON.——Octavo. 240 Pages. Cuts. 1706.

N. B. A Second Edition was printed in the Year 1726.

3. The Method of Fluxions, both Direct and Inverſe. The former being a Tranſlation from the celebrated Marquis *De L'Hospital's* Analyſe des Infinements Petits; and the latter ſupply'd by the Tranſlator, E. STONE, F. R. S.——Octavo. 450 Pages. Plates. 1730.

4. The Doctrine of Fluxions, founded on Sir *Isaac Newton's* Method, publiſhed by himſelf in his Tracts upon the Quadrature of Curves. By JAMES HODGSON, F. R. S. and Maſter of the Royal Mathematical School in Chriſt's Hoſpital.—— Quarto. 452 Pages. Cuts. 1736.

N.B. The Title-Page was reprinted in 1756; and, again, in 1758.

5. The Method of Fluxions and Infinite Series; with it's Application to the Geometry of Curve-Lines. By the Inventor Sir *Isaac Newton*, K[t]. late Preſident of the Royal Society. Tranſlated from the Author's Latin Original, not yet made public. To which is ſubjoined, a Perpetual Com-

ment upon the whole Work. By JOHN COLSON, M. A. and F. R. S.——Quarto. 339 Pages. Cuts. 1736.

N. B. The ſame Piece was tranſlated by another Hand; and publiſhed, without a Comment, in the Year 1737. Octavo. 189 Pages. Cuts.

*_** The Original was written in the Year 1671; but founded on a ſmaller manuſcript Tract compoſed in *November*, 1666; in which the great Inventor uſed the ſame Method of noting the Fluxions of variable Quantities as that which he afterwards generally followed, that is, *Pointing*.

6. A Mathematical Treatiſe: Containing a Syſtem of Conic-Sections; with the Doctrine of Fluxions and Fluents, applied to various Subjects; viz. to the finding the Maximums and Minimums of Quantities; Radii of Evolution, Refraction, Reflection; ſuperficial and ſolid Contents of curvilinear Figures; Rectification of Curve-lines; Centers of Gravity, Oſcillation and Percuſſion: as alſo, to the Reſolution of a ſelect Collection of the moſt uſeful, and many new, Phyſico-Mathematical Problems. By JOHN MULLER.——Quarto. 227 Pages. Plates. 1736.

7. The Doctrine and Application of Fluxions. Containing (beſides what is common on the Subject) a number of *new* Improvements in the Theory and the Solution of a variety of *new* and very intereſting Problems in different Branches of the Mathematics. By THOMAS SIMPSON, F. R. S.—2 Volumes, Octavo. 576 Pages. Cuts. 1750.

A third edition of this work was publiſhed in 1805, by WILLIAM DAVIS, then editor of the Gentleman's Mathematical Companion, &c. To

this Edition is prefixed an Account of the Author's Life.

*** This Work is, perhaps, not inferior to any on the Subject.

††† This great and penetrating Genius was born *August* the 20th, 1710, and died *May* the 14th, 1761.

8. A Treatise of Fluxions. By COLIN MAC LAURIN, A M. Professor of Mathematics in the University of Edinburgh, and F.R.S.——2 Volumes, Quarto. 754 Pages. Plates. 1742.

*** In this Masterly Work, the Subject is handled agreeable to the Method of reasoning used by the ancient Mathematicians.

††† This celebrated Writer was born in *February*, 1698, and died *June* the 14th, 1746.

A Second Edition of this work was published in 1801, by the late Wm. Davies, then Editor of the Gentleman's Mathematical Companion, and Author of a Compleat Treatise on Land Surveying, &c. To this Edition is prefixed an Account of the Life of the Author, the whole embellished with a Striking Likeness of him, taken from his Bust in the Royal Observatory at Greenwich, by Permission of the Reverend Dr. Maskelyne, Astronomer Royal.

9. The Doctrine of Fluxions: not only explaining the Elements thereof, but also its application and use in the several parts of Mathematics and Natural Philosophy. By W. EMERSON. The Second Edition, corrected and greatly enlarged.——Octavo. 432 Pages. 1757.

N.B. The Firſt Edition of this elegant and excellent Work was printed in the Year 1743. Octavo. 300 Pages. Plates.

10. Sir *Iſaac Newton*'s two Treatiſes, of the Quadrature of Curves and Analyſis by Equations of an infinite number of Terms, explained. Containing the treatiſes themſelves, tranſlated into Engliſh, with a large Commentary. By JOHN STEWART, A M Profeſſor of Mathematics in the Mariſhal College and Univerſity of Aberdeen.—— Quarto. 479 Pages. Cuts. 1745.

‡‡‡ The Analyſis by Equations was firſt written before the Year 1669; and the Quadrature of Curves before the Year 1676: But, the laſt Scholium in the Quadratures was added but juſt before the Tract was publiſhed, which was by the great Author himſelf in the Year 1704. The Analyſis was firſt printed 1711.

11. The Method of Fluxions applied to a ſelect number of uſeful Problems. By NICHOLAS SAUNDERSON, L L.D. Late Profeſſor of Mathematics in the Univerſity of Cambridge.——Octavo. 309 Pages. Plates. 1756.

12. A Treatiſe of Fluxions. By ISRAEL LYONS, Junior.——Octavo. 269 Pages. Plates. 1758.

N. B. All the before-mentioned books may be had of Anne Davis, No. 2, Albion Buildings, Alderſgate Street, London.

AN

EXPLANATION

OF

FLUXIONS,

IN A

SHORT ESSAY

ON THE

THEORY.

Quicquid præcipies, esto brevis. HOR.

LONDON:

PRINTED FOR W. INNYS, AT THE WEST END OF ST. PAUL'S,
1741.

N 4

TO

THE READER.

I WAS induced, from the many Diſputes concerning Sir ISAAC NEWTON's Method of Fluxions, to try if that moſt uſeful and noble Kind of Inveſtigation might not be eſtabliſh'd upon more obvious Principles. This gave Birth to the following Eſſay; which, therefore, you are deſired to conſider as an Explanation of the Doctrine itſelf, and not of Sir ISAAC's Manner of delivering it. About that I don't mean, nor pretend to take a Part in any Controverſy. It was, doubtleſs, agreeable to our GREAT AUTHOR's unbounded Invention and Diſcernment: but, I preſume, a more familiar Demonſtration and Phraſe will neither be unacceptable to you, nor at all derogatory to the Merit of HIS Performance, whilſt they tend to confirm and elucidate the very ſame Truths.

Be pleaſed to remark, in the following Pages, with the greateſt Care, that Fluxions are not Quantities actually generated, but exiſting in *Poſſe*; ſuch as would be generated in the ſame invariable Portion of Time.

AN

ESSAY

ON THE

THEORY OF FLUXIONS.

THE Productions of an exalted Genius are very liable to Misconstruction and Cavil, as the Subject is often clouded with some natural Intricacy. Hence, particular Illustrations and easier Methods of Proof become requisite; but it is a Truth, not enough attended to perhaps, that these Devices are not the proper Task of ablest Pens. Such Inquirers, too nearly resembling the Author, never feel all the Weight of a pressing Difficulty: besides a quick Conception is still apt to cause an unusual Conciseness; which, no doubt, must obscure the Sense to many who perceive not but by an easy Chain of Consequences.

This seems a reasonable Apology for my attempting to explain the Theory of Fluxions, that important Doctrine. What I can offer may be better adapted to give general Satisfaction, than are the lofty Essays of eminent Mathematics. Such,

however, is my encouragement to the great Undertaking, and so are my Hopes of Success founded: if I should fail; IN MAGNIS VOLUISSE, SAT EST.

General DEFINITION.

The word FLUXION properly apply'd always supposes the Generation of some Quantity (term'd Fluent or Flowing Quantity) with an equable, accelerated, or retarded Velocity, and is ITSELF the Quantity which MIGHT be UNIFORMLY generated, in A CONSTANT PORTION OF TIME, with the Amount or Remainder of THAT Velocity, at the Instant of finding SUCH FLUXION.

ILLUSTRATION.

Suppose a Point D to move from A with any Velocity, *viz.* equable, accelerated, or retarded. And, at the Instant D leaves A, let EF be taken equal to the Line which MIGHT be generated in a CONSTANT PORTION OF TIME, *mn*, UNIFORMLY, with the Velocity of D at that Instant; for then is EF the Fluxion at A. Now, if AD be generated with an equable Velocity, or, in other Words, if the Velocity at A be the Velocity in every Point of AD, it is plain EF is a constant Quantity, and of course can have no Fluxion. And, on the other hand, when AD is generated with an accelerated or a retarded Velocity, let the Line EF be so increased or decreased by the Motion of a Point F, whilst D moves from A, that this increasing or decreasing EF may always equal the Distance which

Fig. 1.

MIGHT BE UNIFORMLY defcrib'd in THAT CONSTANT PORTION OF TIME, *m n*, with the Amount or Remainder of Velocity at D; for thus is EF ftill the Fluxion of AD *(per Defin.)* tho' a variable Quantity producing its proper Fluxion GH. This is determin'd as EF was at firft, *viz.* take GH equal to the Diftance which MIGHT be UNIFORMLY defcribed in the faid CONSTANT PORTION OF TIME *mn*, with the Velocity of F, whereby EF was increas'd or decreas'd when D left A. It is likewife call'd the fecond Fluxion of AD, as being Fluxion to a Quantity that is the Fluxion of AD. Again, if the original or given Motion wherewith AD is generated be fuch, that GH is alfo variable; let IK be taken equal to the UNIFORM Space which MIGHT be generated in *mn*, with the Velocity of H tending to increafe or decreafe GH, whilft F began to increafe or decreafe EF upon D's leaving A; for fo, IK is the firft Eluxion of GH, the fecond of EF, and the third Fluxion of AD. And, as D advances, the fluxionary Lines, before affumed, increafe or decreafe by the Motions of their refpective Points F, H, K, that in all Pofitions of D they are ftill the firft, fecond, and third Fluxions of AD.—Laftly, it is not difficult to conceive fuch an Acceleration in D, that neither EF, GH, IK, nor any other fucceeding Quantity fhall be UNIFORMLY increas'd or diminifh'd by the Motion of its regulating point F, H, K, *&c.* whence, there will moft manifeftly arife a Progreffion of Fluxions *in Infinitum.* The Defcent of heavy Bodies, accurately confider'd, according to the true Theory of Gravity, affords this infinite Progreffion. Their Defcent by an uniform Gravity no lefs evinces the Limitation of the Orders of

Fluxions: for, if D be uniformly accelerated. GH (the second Fluxion of AD) is a constant Quantity; seeing, an uniform Impulse on D, must necessarily cause an equable Velocity in F, rend'ring GH CONSTANT.

To illustrate the same in plane Figures; let ABC be any triangular or curvilineal Space generated by the Right-Line BD moving uniformly and parallel to itself over the two given or immoveable *Fig.* 2. Lines AL, ACW. B*b* (= AR = RT. = TO) is the uniform Distance describ'd by the Point B in any CONSTANT PORTION of TIME *mn*; that is, B*b* (= AR or RT = TO) is the Fluxion of AB. So, CE being parallel to AB and AP always equal to BC, the Rectangle C*b* or AQ will be the Fluxion of the Area ABC. Now, since the Line PQ keeps moving from AR, and the Rectangle AQ continually increasing as the Ordinate BC increases assume the Rectangle RV equal to the Space that MIGHT be UNIFORMLY describ'd in *mn*, with the Velocity whereby AQ is increas'd in this Position: and then is RV the first Fluxion of AQ, and the second Fluxion of ABC. Again; if the Formation of ACW be such, that the Increase of BC is not equable, RV will be subject to a Variation likewise. Take, therefore, the Rectangle TS equal to the UNIFORM Space which MIGHT be produced in Time *mn*, with the Velocity of RV's Increase. This TS is evidently the first Fluxion of RV, the second of AQ, and the third Fluxion of ABC: and after the same manner are all Fluxions, whether of Lines, Surfaces, or Solids, to be consider'd. But, be it ever remember'd that the Velocity with which a Quantity is said to be generated, is not esteem'd the Velocity of any of its particular Parts

or Terms, but the Celerity or Degree of Swiftneſs wherewith the Magnitude of that Quantity is changed.

Corollary.

Hence, it will appear that the firſt Fluxions of Quantities are as the Velocities with which thoſe Quantities are increaſ'd; that ſecond Fluxions are as the Increaſe or Decreaſe of ſuch Velocities; and that by ſecond, third, fourth, *&c.* Fluxions are meant Fluxions, whoſe Fluents are themſelves Fluxions to other propoſed Quantities; and the manner of conſidering, and determining them is the very ſame as tho' they were firſt Fluxions, they being actually ſo to the Quantities from which they are immediately derived.

Theſe Particulars duly weigh'd will (I hope) remove all the Difficulties and ſeeming Inconſiſtencies ſo often complain'd of in a Progreſſion of Fluxions. There is no eſſential Difference amongſt them: the Proceſs is only the more tedious the higher we go. I ſhall therefore proceed to lay down a Lemma and Propoſition, and from thence endeavour to deduce the neceſſary Rules for determining all Orders of Fluxions: but firſt of all obſerve the following

Notation.

$\dot{y}$ ſtands for the firſt Fluxion of y; $\ddot{y}$ for the ſecond Fluxion of y, and firſt Fluxion of $\dot{y}$; $\dddot{y}$ for the third Fluxion of y, the ſecond Fluxion of $\dot{y}$, and the firſt Fluxion of $\ddot{y}$, *&c.* $\dot{z}$ ſtands for the firſt

Fluxion of $\dot{z}$; $\ddot{z}$ for the second Fluxion of z, and the first Fluxion of $\dot{z}$, &c. and so on, for any other.

Lemma.

The Fluxion of the Area ABC, whether triangular or curvilineal, is the Rectangle $\dot{x}y$.

Suppose a Body B to move from A towards F, and to send forth a Ray y always perpendicular to
Fig. AF, and lengthening, as the Body approaches F;
3. so as, by its Extreme C, to describe the Curve or right Line AC: And, at any proposed Position BC, conceive y to become CONSTANT, while the Body moves UNIFORMLY any CONSTANT TIME *mn*, with the Velocity at B, over the Distance $\dot{x}$ or BD; for then will y in the Time *mn* UNIFORMLY generate the Rectangle $\dot{x}y$, which Rectangle is plainly the Fluxion of ABC in this Position (*per Definit.*)

Scholium.

It has been commonly objected to the Accuracy of Fluxions, that the Trapezium or curvilineal Space BC*de*D, not the Rectangle $\dot{x}y$, is the Fluxion geometrically exact. But, this objection is built, I apprehend, upon a false Idea of the
Fig. Thing. It supposes a Fluxion a COMPLETE Part
3. of a flowing Quantity, and an INFINITY of Fluxions to constitute the flowing Quantity, which are Mistakes (*per* Definition and Lemma.) The Area BC*de*D is the Increment; the Space that WOULD have been generated in TIME *mn* with y variable;

and indeed if $\dot{x}$ be imagined infinitely little, an Infinity of Increments may conſtitute the Area ABC. But in Fluxions, our reaſoning is quite different: a Fluxion can no more be called a Part of the Fluent, than an Effect a Part of the Cauſe. For Inſtance; from the Fluxion given we know the Fluent, and *vice versa*, juſt as when a Cauſe is known to produce a certain Effect, we can infer the ONE from a Knowledge of the OTHER. Of the ſame Kind is the common Objection againſt the Fluxion of a Curve, (*Fig.* 4.) that $\sqrt{\dot{x}^2 + \dot{y}^2}$, not expreſſing a Part of the Curve, is not accurately the Fluxion. But it is accurately ſo, for $\sqrt{\dot{x}^2 + \dot{y}^2} = \dot{z}$ a ſtraight Line which WOULD be deſcrib'd in the TIME *mn* UNIFORMLY, with the Velocity of $\dot{x}$ and $\dot{y}$ compounded, which are the Amount of Velocity wherewith the Curve is generated at that Inſtant.

PROPOSITION.

The Fluxion of a Rectangle xy is $\dot{x}y + \dot{y}x$.

Let two Bodies, B.C. move from A the ſame Moment towards G and H, and carry along with them the perpendicular right Lines BF.CE. The Path AP of their Point of Interſection P varies according to the Relation of the Velocities B.C; but ſtill there are generated with a variable Celerity (like ABC in the *Lem.*) two Areas ABP.ACP, *Fig.* 5. which together are always equal to the Rectangle BACDB. Here then, as $y\dot{x} =$ the Fluxion of ABP, and $x\dot{y} =$ Fluxion of ACP, (*per Lem.*) and as ABP + ACP = the Rectangle ABCDB $= xy$; and equal Fluents have equal Fluxions, $y\dot{x} + x\dot{y}$ is conſequently = Fluxion of xy. *Q. E. D.*

Corollary I.

If AC = AB, the Figure generated is a Square. Then is x^2 or y^2 the Fluent, and $2x\dot{x}$ or $2y\dot{y}$ the Fluxion.

Corol. II.

Suppose AC = second Power of BA. Then $xy = x^3$, and the Fluxion is $3xx\dot{x}$. For $y = xx . \dot{y} = 2x\dot{x} . \dot{y}x = 2xx\dot{x} . y\dot{x} = xx\dot{x}$ ∴ $\dot{y}x + \dot{x}y = 3xx\dot{x}$ the Fluxion of x^3.

Corol. III.

And universally; let $xy = x^m$, and by a like Method of Investigation, the Fluxion will be found $mx^{m-1}\dot{x}$.

From this Proposition, and its Corollaries, I shall now deduce the Practical Rules for finding the Fluxions of variable Quantities multiply'd together; of Fractions, and of Powers. Examples in the higher Orders of Fluxions will follow.

These Rules are laid down in Mr. *Simpson's* Treatise of Fluxions, thus, *viz.* To find the Fluxion of the Product of several Quantities drawn into each other,

RULE.

Multiply the Fluxion of each particular Quantity

by the Product of the rest of the Quantities, and the Sum of the Products arising from those Multiplications will be the Fluxion sought.

For, by the Proposition, $x\dot{y} + y\dot{x} =$ Fluxion of xy; and if the Fluxion of xyz be sought, put (as Mr. *Simpson* has done) $v = xy \therefore \dot{v} = x\dot{y} + y\dot{x}$, and the Fluxion of vz or xyz will be $v\dot{z} + z\dot{v}$, but $\dot{v} = x\dot{y} + y\dot{x}$ and $v = xy \therefore$ substituting these Values in $v\dot{z} + z\dot{v}$, it will become $zx\dot{y} + zy\dot{x} + xy\dot{z}$ $=$ the Fluxion of xyz.

To find the Fluxion of a Fraction,

RULE.

From the Fluxion of the Numerator drawn into the Denominator, take the Fluxion of the Denominator drawn into the Numerator, and divide the whole by the Square of the Denominator.

For (following the same Author) by putting $v = \frac{x}{y}$ we have $vy = x$, and $v\dot{y} + y\dot{v} = \dot{x}$ (*per Prop.*)

$$\therefore \dot{v}y = \dot{x} - \dot{y}v \left(v = \frac{x}{y}\right) = \dot{x} - \frac{\dot{y}x}{y} = \frac{y\dot{x} - x\dot{y}}{y}$$

$$\therefore \dot{v} = \frac{y\dot{x} - x\dot{y}}{yy} \text{ the Fluxion of } \frac{x}{y}.$$

To find the Fluxion of any Power of a variable Quantity.

RULE.

Multiply the Exponent of the given Power by

the Fluxion of the Root, and that product by the Power of the Root, whose index is one less than that of the given Power.

This follows immediately from the Third Corollary, and is indeed no more than $mx^{m-1}\dot{x}$ in Words.

An Example of the First Rule.

The Fluxion of $x\dot{y} + y\dot{x}$, or second Fluxion of xy is $2\dot{x}\dot{y} + \ddot{y}x + \ddot{x}y$, when $\dot{x}$ and $\dot{y}$ are both variable; $2\dot{x}\dot{y} + \ddot{y}x$, or $2\dot{x}\dot{y} + \ddot{x}y$ when only one of them is variable, and $2\dot{x}\dot{y}$ alone if neither be variable.

An Example of the Second Rule.

The Fluxion of $\frac{y\dot{x} - x\dot{y}}{yy}$ *Viz.* the Second Fluxion of $\frac{x}{y}$, is $\frac{2x\dot{y}\dot{y} - 2y\dot{y}\dot{x} + yy\ddot{x} - xy\ddot{y}}{y^3}$

Examples of the Third Rule.

$2\dot{x}\dot{x} + 2x\ddot{x}$ is $=$ the Fluxion of $2x\dot{x}$. $6x\dot{x}\dot{x} + 3xx\ddot{x} =$ the Fluxion of $3xx\dot{x}$; for, putting $xx = y$, we have $2x\dot{x} = \dot{y}$, and $2\dot{x}\dot{x} + 2x\ddot{x} = \ddot{y}$; (*per* Cor. 2. and Notation) ∴ by substituting these Values of y, $\dot{y}$ and $\ddot{y}$ in the first Example, $2\dot{x}\dot{y} + \ddot{y}x + \ddot{x}y$ becomes $6x\dot{x}\dot{x} + 3xx\ddot{x}$, the second Fluxion of x^3. And universally the second Fluxion of x^m is $m - 1 \times mx^{m-2}\dot{x}^2 + mx^{m-1}\ddot{x}$. The Fluxion of this again *viz.* the third Fluxion of x^m is $m - 2 \times m - 1$

$\times\ m\,x^{m-3}\,\dot{x}^{3} + 3m \times m - 1 \times m x^{m-2}\,\ddot{x}\dot{x} + m x^{m-1}\,\dddot{x}$. Theſe Expreſſions are eaſily diverſify'd upon ſuppoſing x or $\dot{y}$ or both conſtant: and the Method being the ſame for all Orders however high we go, I think it ſuperfluous and unneceſſary to enlarge.

So here I ſhall conclude, preſuming that This may ſuffice to give, in general, an accurate Idea of the Doctrine of Fluxions; which is all I aimed at. The Application to Phyſics and Mathematics is foreign to my Purpoſe, and not ſuited to a ſlender Skill and Experience in theſe Studies. I might perhaps with Juſtice enough add too, a Performance of that Kind is ſcarce wanted; for, what more elegant Examples and Solutions can we expect or deſire than are extant in the Works of *our own Mathematicians?* My Buſineſs was only to pave the Way a little: but, *Eſt quadam prodire tenus.*

FINIS.

BOOKS

Printed for ANNE DAVIS,

NO. 2, ALBION BUILDINGS, ALDERSGATE STREET;

And sold by Sherwood and Co. and Wm. Baynes, Paternoster Row; Lackington and Co. Finsbury Square; J Nunn, Great Queen Street, Lincoln's Inn Fields; and J. S. Dickson, 18, Ivy Lane, Paternoster Row.

On the 9th November, (continued annually,) will be published price 2s. stitched.

The GENTLEMAN'S MATHEMATICAL COMPANION; containing answers to the last year's Enigmas, Rebuses, Charades, Queries, and Questions, also new Enigmas, Rebusses, Charades, Queries, and Questions, proposed to be answered the year following: to which will be added, some original Tract, or Valuable Selections, taken from Authors of known celebrity.

Persons possessing imperfect sets of this work, and desirous of completing them, are requested to apply immediately for the deficiences. A few sets remaining unsold, may be had at the lately advanced price of 2s. per Number.

In the press, and will be published about the same time, a second Edition of

The KEY to BONNYCASTLE's ALGEBRA.

In this Edition all the questions are worked over a fresh, and new or corrected Solutions given to those that are defective; the whole carefully revised, and adapted to the last Edition of the Algebra, by a Person of known celebrity in the Mathematical Sciences.

Speedily will be published, a fifth Edition of

A complete TREATISE of LAND SURVEYING by the Chain, Cross, and Offset Staffs, shewing the various methods of Measuring, Mapping, and Dividing Land, &c. with an Appendix, containing the Measurement of Timber, Hay, Marl-pits, &c. &c. with 8 folding Plates By Wm. Davis, late Editor of the Gentleman's Mathematical Companion. To this Edition will be added a portrait of the Author.

By the same Author,

A short but comprehensive TREATISE on the GLOBES,

the SOLAR SYSTEM, &c. in which the New Discoveries are introduced, 12mo. neatly bound, 2s. 6d.

A KEY to BONNYCASTLE's MENSURATION, in which all the Questions unanswered are solved in a clear and comprehensive Manner, neatly bound, 12mo, 3s.

A KEY to BONNYCASTLE's ARITHMETIC, containing Solutions to the Questions left unanswered in that useful Work, with the References as they stand in the last Edition. Printed uniformly with the Keys to Bonnycastle's Algebra and Mensuration, neatly bound, price 4s. 6d.

AN ELEMENTARY TREATISE of FLUXIONS, by COLIN MACLAURIN, F. R. S.; in 2 large octavo vols. fine paper, with 42 copper plates, boards, 1l. 7s.

A TREATISE of FLUXIONS, originally in 2 vols. by THOMAS SIMPSON, F. R. S.; now brought into one, uniform with the above, boards, 16s.

NEWTON'S (Sir Isaac) PRINCIPIÆ, by MOTTE, in which is embodied the System of the World, and Emerson's Comment, 3 vols. 8vo, boards, 1l. 10s

FENNING'S EASY GUIDE to PRACTICAL ARITHMETIC, with great Improvements and Additions.

On the 13th May, 1803, was published, price 4s. 6d bound, being the 7th, and the 2d Edition by the late Wm. Davis,

FENNING'S ALGEBRAIST'S COMPANION; or, a new and easy GUIDE to ALGEBRA, introduced by the Doctrine of Vulgar Fraction, designed for the use of schools, and for such who by dint of their own application would become acquainted with the Rudiments of that noble Science. Illustrated with a variety of Numerical and Literal examples, with an Appendix on the Rudiments of Quadratic Equations, Extraction of Roots, &c. with a Supplement, by the said W. Davis, containing thirty-eight Select Problems, with their Solutions, which illustrate the Method of expunging several unknown quantities, with Solutions of Equations of the higher order, and method of infinite Series, &c.

N. B To this Edition is added a SUPPLEMENT of a former Edition by the Reverend J. HELLINS, containing the method of exterminating two or more unknown quantities, out of an Equation, some further observations on Quadratics, &c.

N. B. This work was began and prepared for the press by the late Wm. Davis, and the press has been superintended by a Person well versed in these Sciences.

*** The last mentioned five works were Edited by the late William Davis.

A TREATISE on SPHERICAL GEOMETRY, containing its Fundamental Properties; the Doctrine of its Loci; the Maxima and Minima of Spherical Lines, and Areas; with an Application of these Elements to a variety of Problems. Plates, 8vo, price 7s. boards. By JOHN HOWARD.

N. B. Only a few Copies of this valuable work remain unsold.

Shortly will be published,

A Catalogue of scarce Mathematical and Philosophical Books selected from the best Authors, and most of them out of print; amongst which are the following: Atwood on Rectilinear Motion.—Emerson's Works, complete.—Simpson's Works, complete.—Simson's (Robert) Opera Quædam Reliquia.—Apollonii Pergœli Locorum Planorum.—Newton's Works, by Dr. Horseley.—Ditto Principia Mathematica, a Jacquier, 4 tom.—Ditto Arithmetica Universalis, by Castilioni, 2 tom.—Newton's Quadrature of Curves, by Stewart.—The Mathematician.—Turner's Mathematical Exercises.—The Student; Liverpool, 4 Numbers, all ever published.—Complete Sets of Gentlemen's and Ladies' Diaries, Palladiums, and Ephemeries.—Whiting's Poetical and Mathematical Delights.—Scientific Receptacle.—Stockton Bee, being a collection of Poetry, Queries, Mathematical Questions, &c. &c.

Philosophical Transactions, 90 vols.—Encyclopædia Britannica, 20 vols. bds.—Euler's Works, Foreign and English editions.--Demoivre's Miscellanea Analitica.--Ptolemei Planisphœrum.—Ditto de Analemmate.—Hooke's Philosophical Tracts.—Moxon's Mechanic Exercises, 3 vols.—Ricciole Almagestum, Astronomium Veterum.—Smith's Optics.—Rioz's Tables of Astronomy and Navigation, 1805.—Parkinson's Mechanics and Hydrostatics.—La Place's Theorie de Planetes.—Hatton, on Clock and Watch-making.—Taylor's (Brook) Linear Perspective, 8vo.—La Grange's Theorie de Fonction.—Banks, on Mills.—Landen's Mathematical Memoirs.—Brownrig, on making common Salt.—European Magazine, from its commencement.—Le Gendre de Geometre.—Malcolm's System of Arithmetic.—Vince's Astronomy.—Holliday's Miscellanea Curiosa Mathematica.—Ditto Syntagma Mathesios.—Maseres's Scriptores Logarithmici, 6 vols.—Sanderson's Algebra, &c.

Printed by R. WILKS,
89, Chancery Lane.

Plate I.

Fig. 1 to 8.

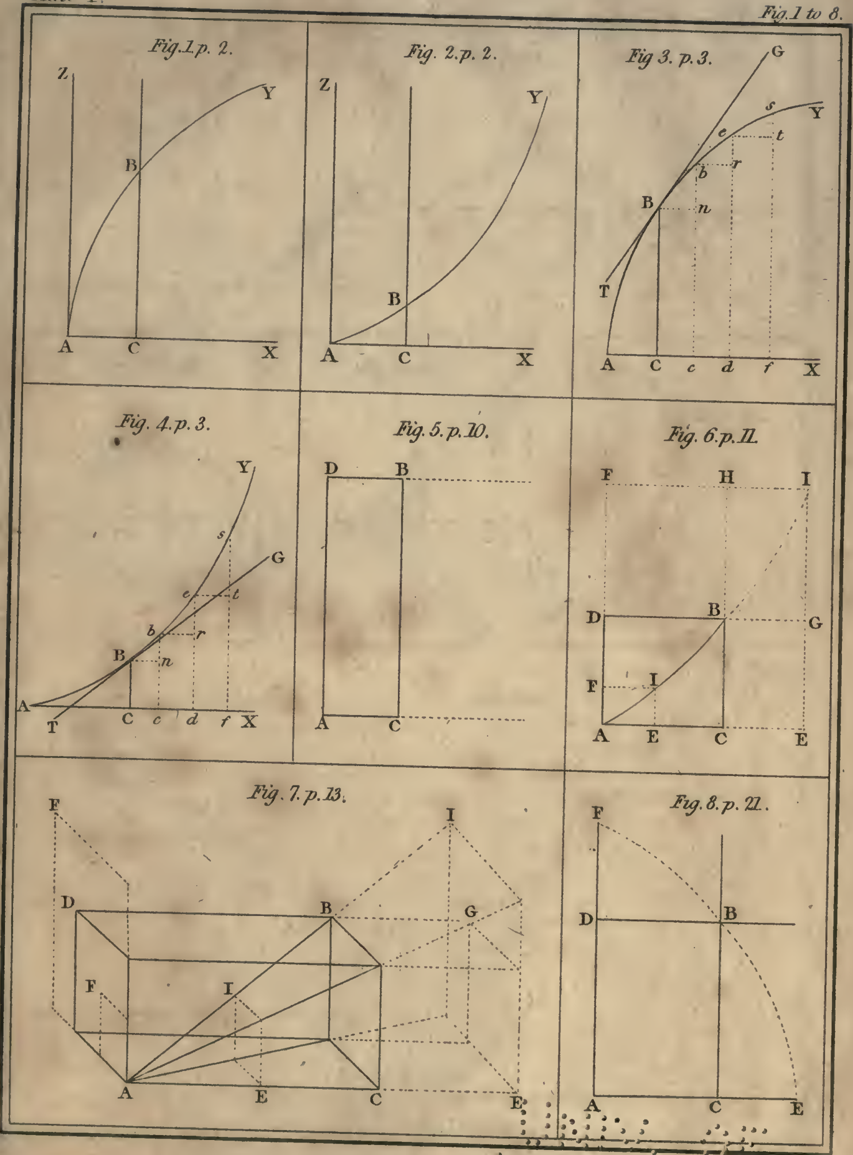


Fig. 9. p. 22.

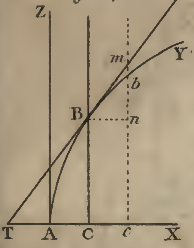


Fig. 10. p. 22.

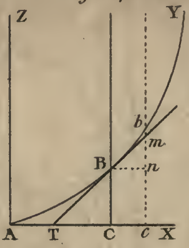


Fig. 11. p. 25.

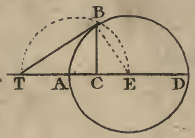


Fig. 12. p. 26.

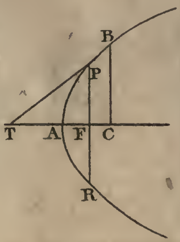


Fig. 13. p. 26.

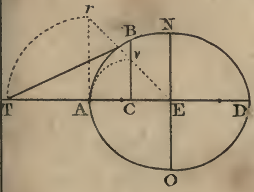


Fig. 14. p. 27.

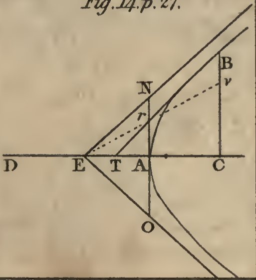


Fig. 15. p. 29.

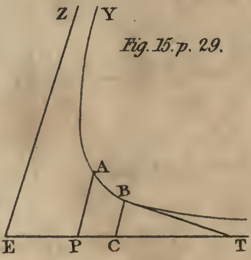


Fig. 16. p. 30.

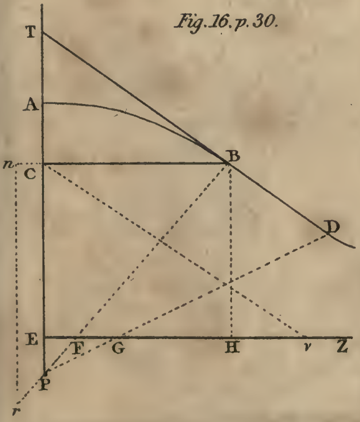


Fig. 17. p. 32.

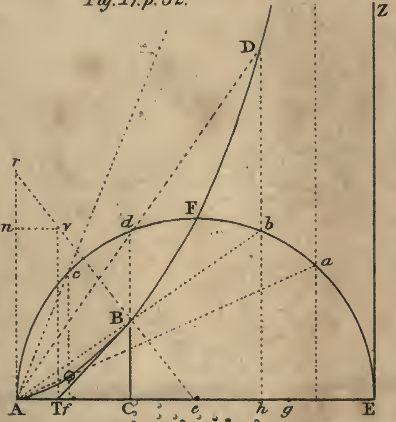


Fig. 18. p. 33.

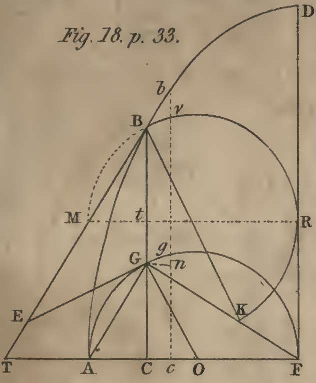


Fig. 19. p. 35.

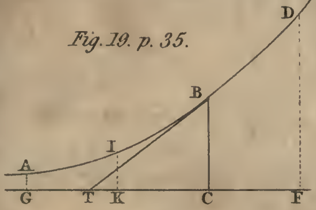


Fig. 20. p. 36.

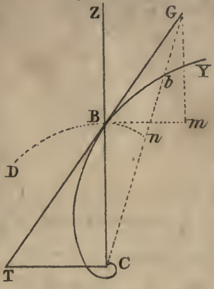


Fig. 21. p. 38.

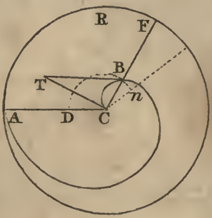


Fig. 22. p. 39.

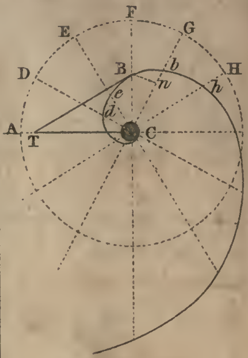


Fig. 23. p. 41.

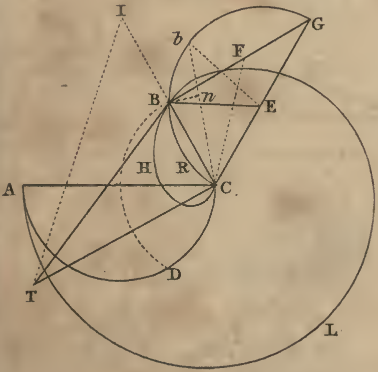


Fig. 24. p. 41.

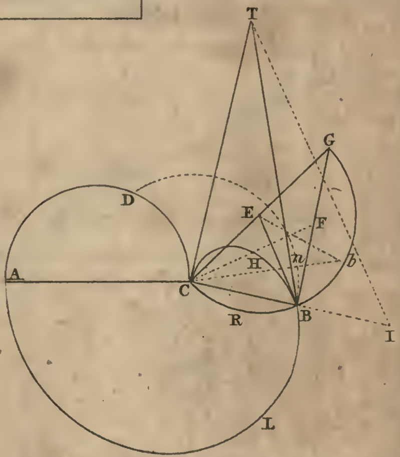


Fig. 25. p. 44.

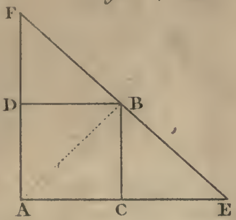


Fig. 26. p. 47.

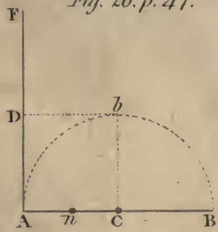


Fig. 27. p. 50.

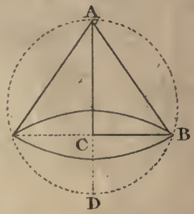


Fig. 29. p. 54.

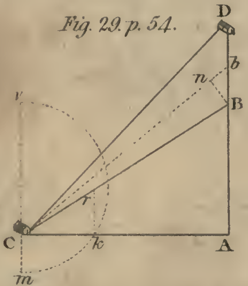


Fig. 30. p. 57.

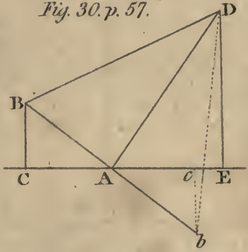


Fig. 31. p. 58.

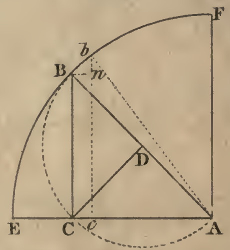


Fig. 28. p. 52.

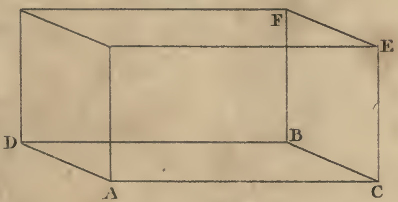


Fig. 32. p. 60.

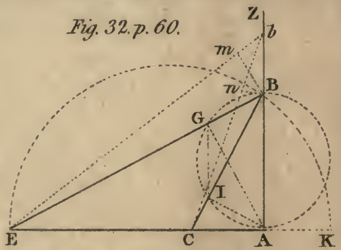


Fig. 33. p. 61.

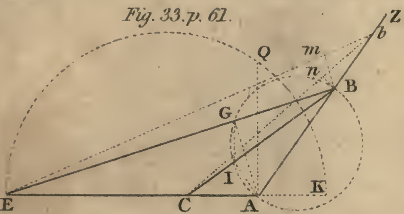


Fig. 34. p. 61.

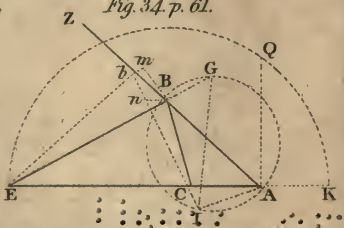


Plate V. Fig. 35 to 38.

Fig. 35. p. 62.

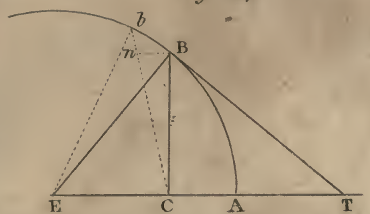


Fig. 36. p. 63.

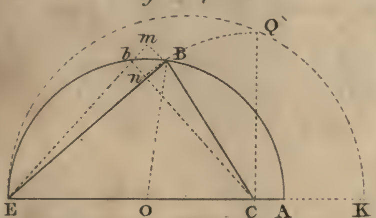


Fig. 37. p. 65.

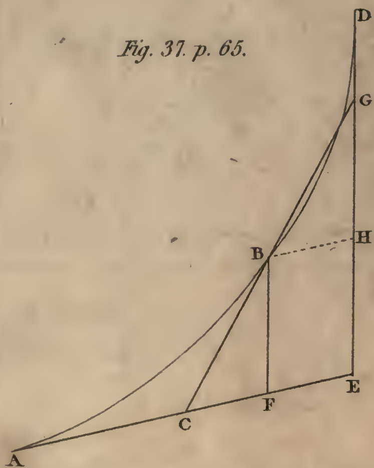


Fig. 38. p. 66.

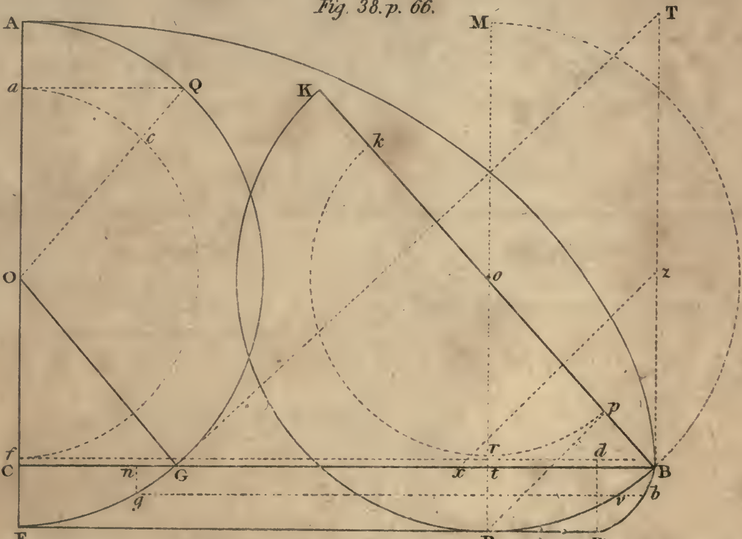


Fig. 39. p. 69.

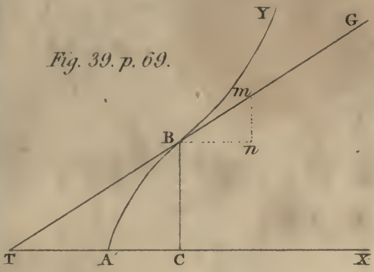


Fig. 40. p. 69.

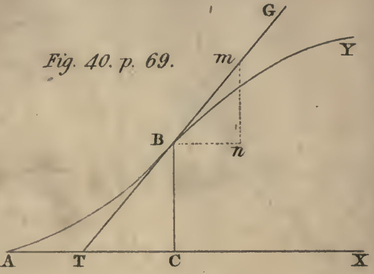


Fig. 41. p. 73.

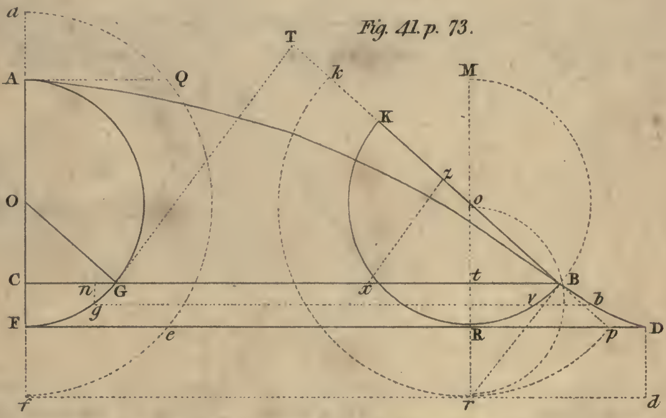


Fig. 42. p. 76.

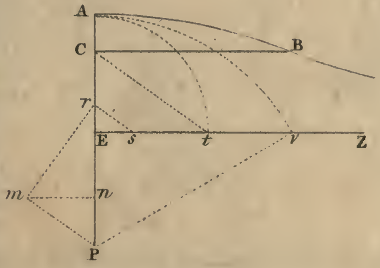


Fig. 43. p. 78.

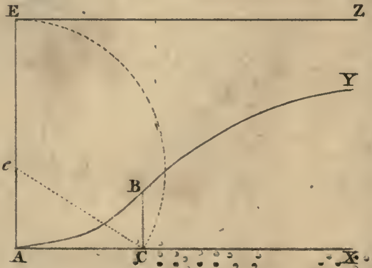


Fig. 44. p. 81.

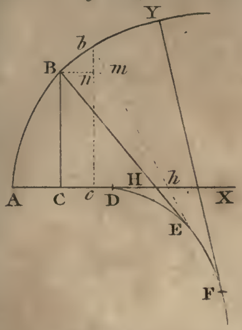


Fig. 45. p. 83.

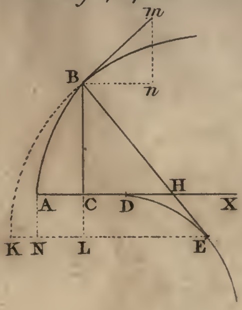


Fig. 46. p. 86.

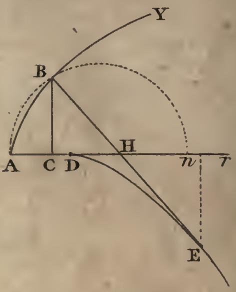


Fig. 47. p. 88.

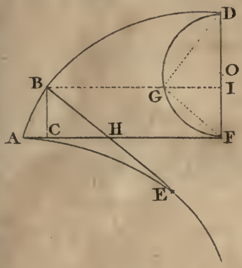


Fig. 48. p. 90.

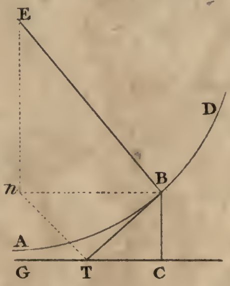


Fig. 49. p. 92.

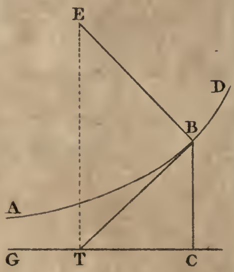


Fig. 50. p. 93.

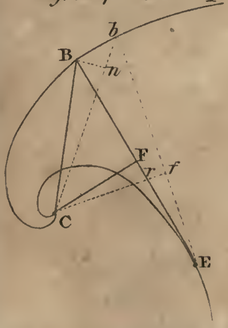


Fig. 51. p. 95.

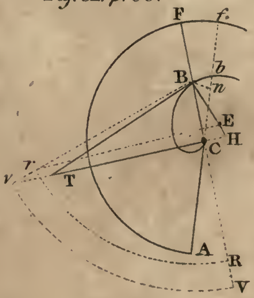


Fig. 52. p. 97.

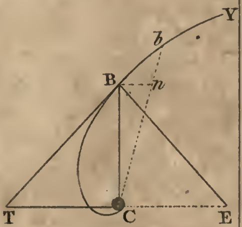


Fig. 53. p. 99.

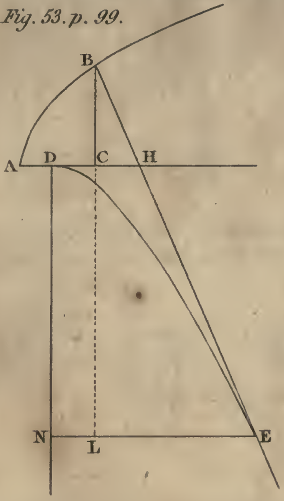


Fig. 54. p. 102.

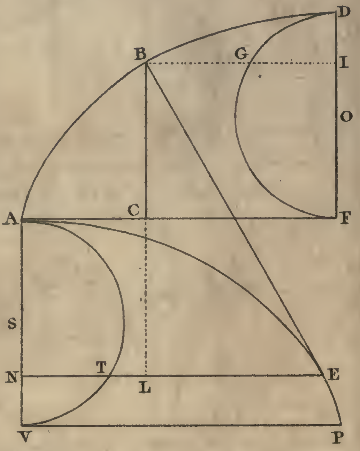


Fig. 55. p. 103.

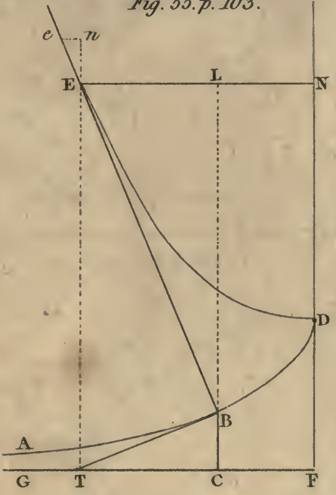


Fig. 56. p. 105.

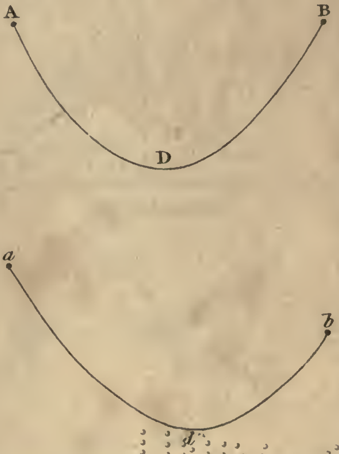


Fig. 57. p. 130.

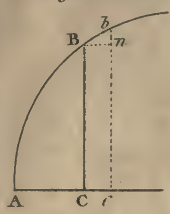


Fig. 58. p. 130.

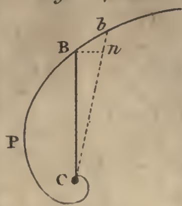


Fig. 59. p. 131.

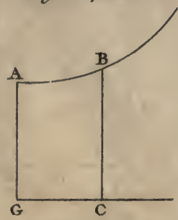


Fig. 60. p. 132.

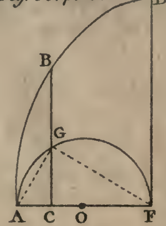


Fig. 61. p. 133.

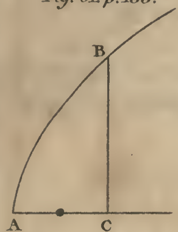


Fig. 62. p. 135.

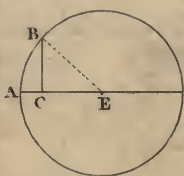


Fig. 63. p. 137, & 143.

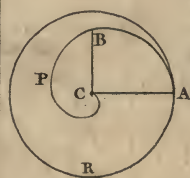


Fig. 64. p. 139.

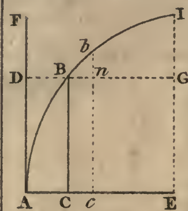


Fig. 65. p. 139.

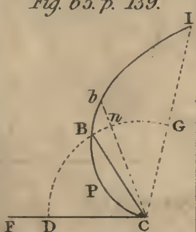


Fig. 66. p. 141.

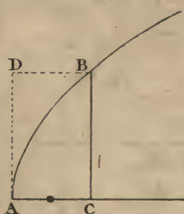


Fig. 67. p. 142.

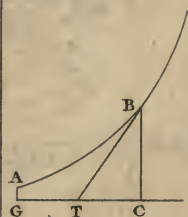


Fig. 68. p. 143.

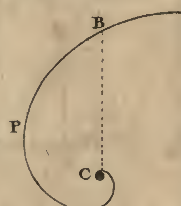


Fig. 69. p. 144.

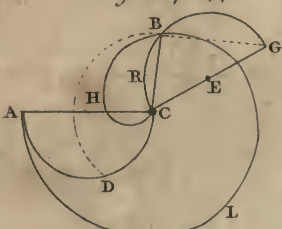


Fig. 70. p. 144.

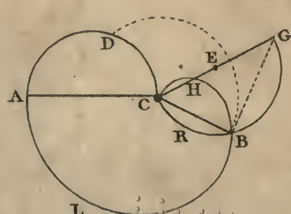


Fig. 71. p. 145.

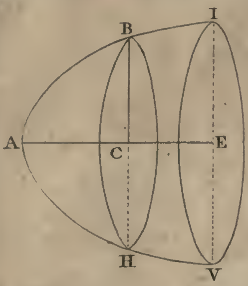


Fig. 72. p. 147.

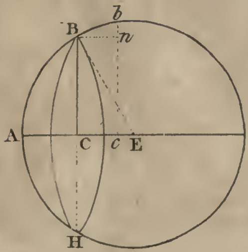


Fig. 73. p. 148.

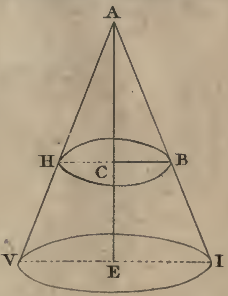


Fig. 74. p. 149. & 152.

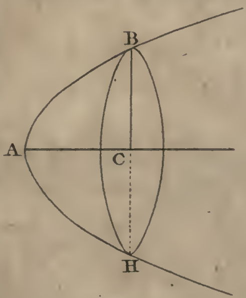


Fig. 75. p. 149.

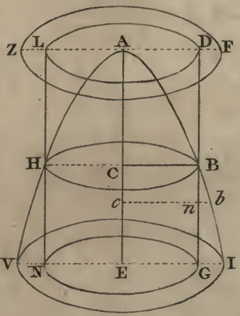


Fig. 78. p. 155.

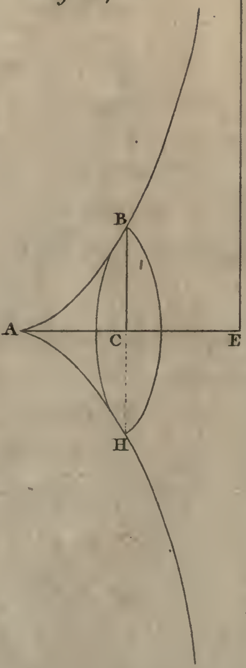


Fig. 76. p. 151.

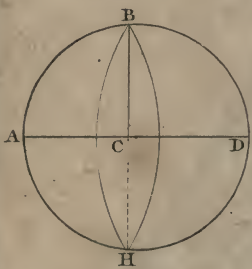


Fig. 77. p. 153.

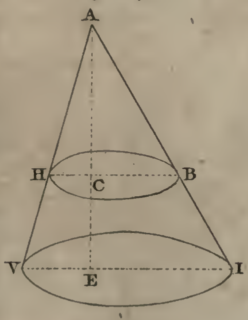


Fig. 79. p. 157.

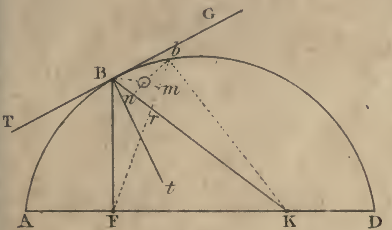


Fig. 80. p. 158.

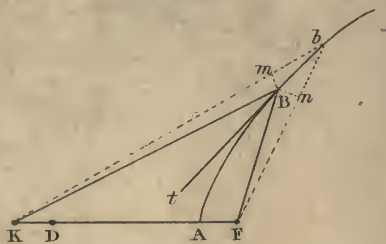


Fig. 81. p. 158.

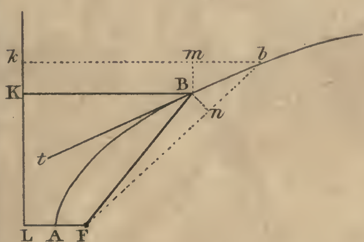


Fig. 82. p. 159.

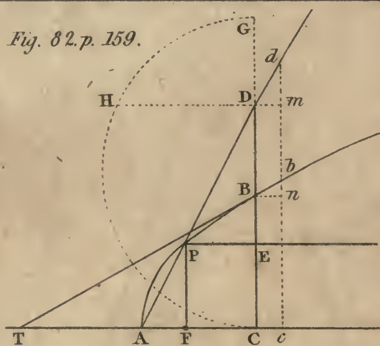


Fig. 83. p. 160.

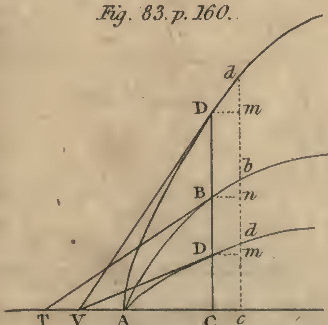
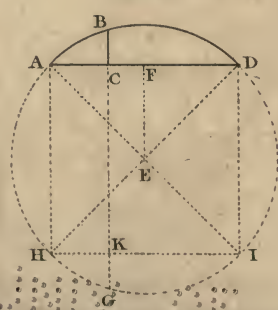


Fig. 84. p. 161.



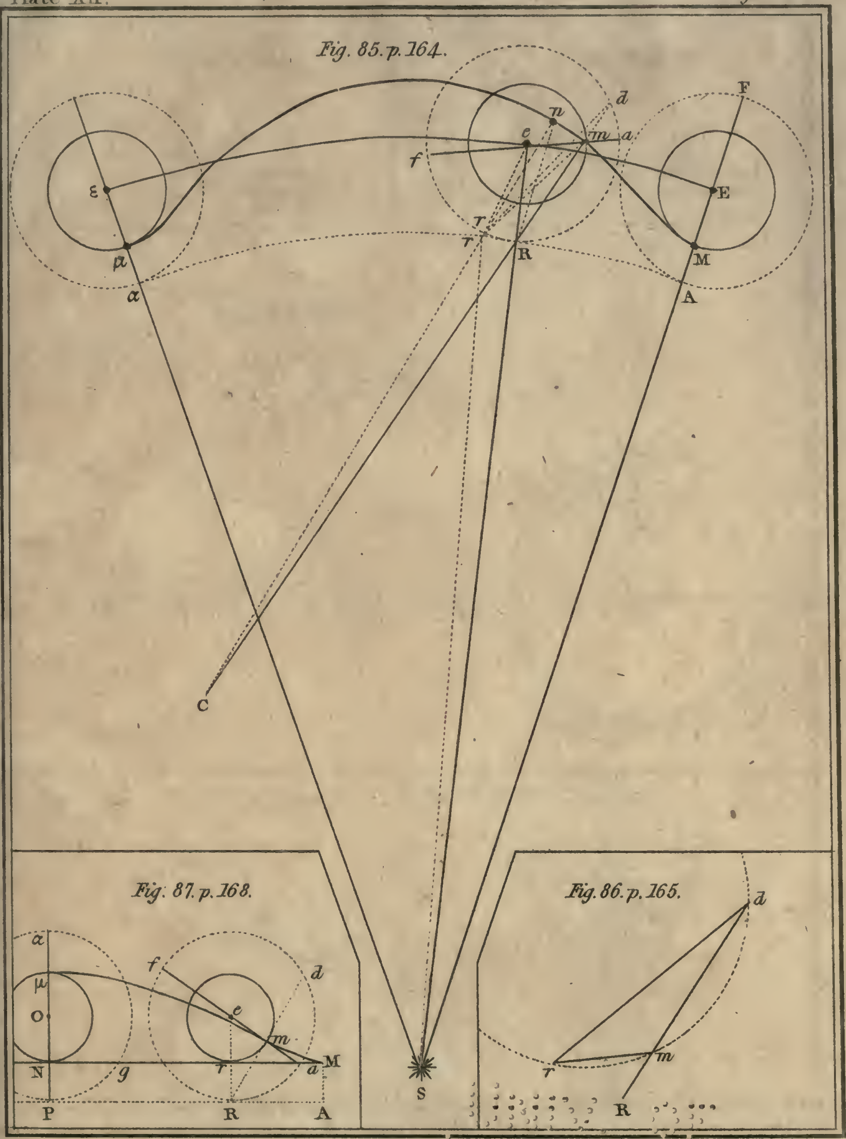


Fig. 85. p. 164.
Fig. 87. p. 168.
Fig. 86. p. 165.

Plate XIII. Fig. 88. to 92.

Fig. 88. p. 169.

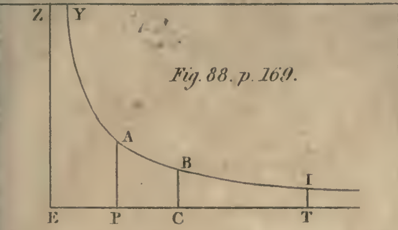


Fig. 89. p. 170.

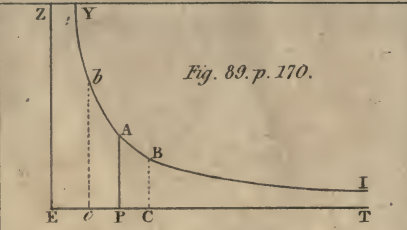


Fig. 90. p. 172.

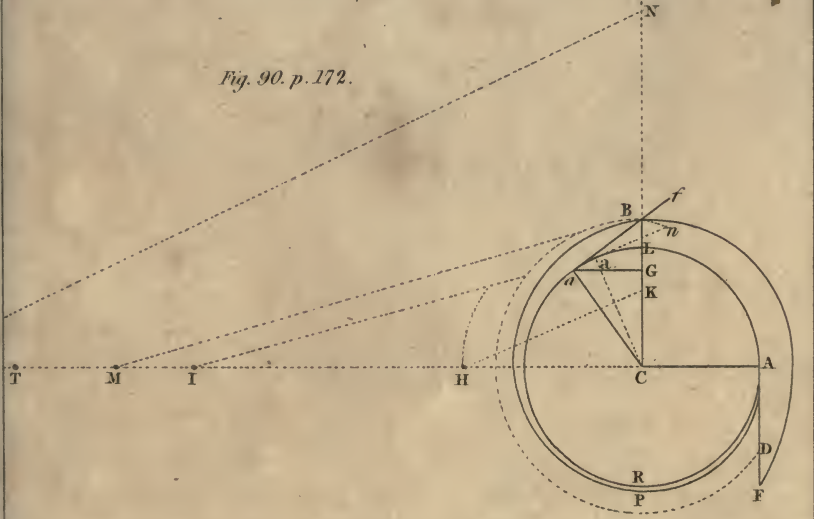


Fig. 91. p. 175.

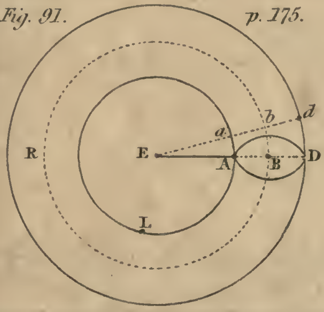


Fig. 92. p. 176.

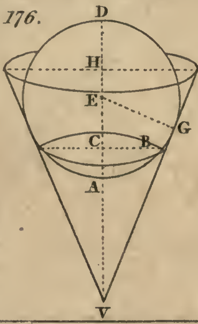


Plate XIV. marked D.